Resonant frequencies of lossy dielectric spheres excited by delta function electric and magnetic sources

R. CHATTERJEE AND GLORY JOHN

Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560 012

Abstract

The problem of a lossy sphere excited in symmetric TM and TE modes by appropriate delta function sources is considered. The resonant frequencies and the amplitude constants involved in the field components have been computed as functions of both permeability and permittivity of the sphere.

The results have been utilised to verify the frequencies of maximum absorption of biological spheres obtained experimentally[14].

Key words : Resonant frequencies, lossy dielectric spheres, delta function, amplitude constants.

1. Introduction

Some work has been done during previous years on dielectric resonators of different shapes, including spheres[1-9]. Resonators made of materials of high dielectric constant and low loss factor lend themselves to a number of applications such as microwave filters, power limiters, and Gunn and transistor oscillators using YIG spheres[16], as well as in dielectric-tuned and temperature-compensated Gunn oscillators, etc. However, their use requires a good knowledge of their resonant frequencies and amplitude constants of field components.

The electromagnetic boundary value problem of the dielectric sphere excited by delta function electric and magnetic sources applied normally across an arbitrary plane has been solved and the possibility of the existence of symmetric as well as unsymmetric TE, TM and hybrid modes has been investigated by Chatterjee[10], leading to the following conclusions:

(i) It is not possible to excite unsymmetric TE and TM modes on the dielectric sphere since the application of boundary value method yields three independent equations with only two unknown amplitude constants.

(ii) It is possible to excite symmetric TE and TM modes, as well as symmetric and unsymmetric hybrid modes on the dielectric sphere.

21

The theoretical investigation in this paper differs from the studies reported so far by other authors[1-9] in the following respects. We have considered lossy spheres having permittivity (ϵ_1) and permeability (μ_1), ranging from small to very large values. We have also evaluated the resonant frequencies of the sphere as a function of a/v_d for several TE_{on} and TM_{on} modes excited by appropriate delta function sources, where $v_d = (1/\mu_1\epsilon_1)^{1/2}$ denotes the intrinsic phase velocity inside the sphere. We have also studied the amplitude constants of the field components, both inside and outside the sphere as functions of a/λ_0 with ϵ_1 and μ_1 as parameters, where λ_0 denotes the free space wavelength outside the sphere. It has also been shown that the resonant frequencies of lossy dielectric spheres can be related to the frequencies of maximum absorption of electromagnetic radiation by biological spheres as reported by Gandhi[14]. The study of the amplitude constants is important as the temperature rise of biological spheres illuminated by electromagnetic radiation depends not only on the resonant frequencies of maximum absorption, but also on the incident radiation energy, which is proportional to the square of the amplitude constants.

2. Electromagnetic boundary-value problem of dielectric sphere excited in TM and TE modes

Fig. 1 shows the geometry of the structure. Spherical polar coordinates r, θ, ϕ are used. A dielectric sphere of radius 'a' and constants ϵ_1, μ_1, σ_1 is embedded in another dielectric medium (free space) of constants ϵ_0, μ_0, σ_0. The sphere is excited in the TM symmetric mode by an excitation electric field $E_u e^{-i\omega t}$ applied in the z direction over an annular ring of width $\triangle s \to 0$ in the plane $z = z_1 = a\cos\theta_1$. E_0 has components
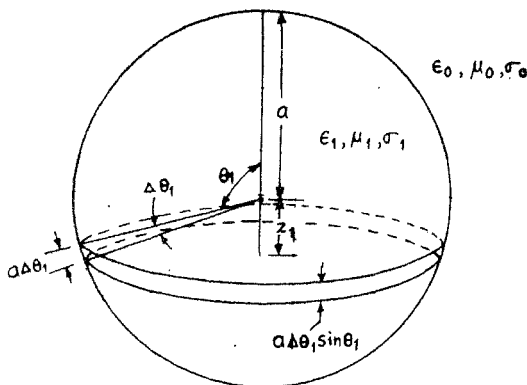

FIG. 1. Geometry of the structure.

$E_{r_0} = E_0 \cos \theta_1$ and $E_{\theta_0} = - E_0 \sin \theta_1$ in the r and θ directions respectively. Expanding the field components E_{r_0} and E_{θ_0} in series of spherical harmonics and assuming harmonic time dependence, we obtain

$$E_{r_0} = \frac{-1}{k_1} \sum_{n=0}^{\infty} n(n+1) D_{on}(r) P_n(\cos\theta) e^{-j\omega t} \tag{1}$$

$$E_{\theta_0} = \frac{-1}{k_1} \sum_{n=0}^{\infty} C_{on}(r) P_n^1(\cos\theta) e^{-j\omega t} \tag{2}$$

where

$$C_{on}(r) = \frac{-k_1(2n+1)}{2\pi n(n+1)} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} E_{\theta_0} P_n^1(\cos\theta) \sin\theta \, d\theta \, d\phi \tag{3}$$

and

$$D_{on}(r) = \frac{-k_1(2n+1)}{2\pi n(n+1)} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} E_{r_0} P_n(\cos\theta) \sin\theta \, d\theta \, d\phi. \tag{4}$$

If E_0 is a δ-function given by

$$E_0 = \frac{-V}{a \triangle \theta \sin\theta_1} \quad \text{for} \quad \theta_1 < \theta < (\theta_1 + \triangle\theta_1) \tag{5}$$

and

$$E_0 = 0 \quad \text{for} \quad \theta < \theta_1 \quad \text{and} \quad \theta > (\theta_1 + \triangle\theta_1). \tag{6}$$

Then

$$C_{on}(a) = \frac{-V}{a} \frac{k_1(2n+1)}{n(n+1)} \sin\theta_1 P_n^1(\cos\theta_1) \tag{7}$$

and

$$D_{on}(a) = \frac{V}{a} k_1 \frac{(2n+1)}{n(n+1)} \cos\theta_1 P_n(\cos\theta_1). \tag{8}$$

The field components inside the sphere are

$$E_r^i = - \sum_{n=0}^{\infty} n(n+1) A_{on} P_n(\cos\theta) \frac{j_n(k_1 r)}{k_1 r} e^{-j\omega t} + E_r. \tag{9}$$

$$E_{\theta}^{i} = - \sum_{n=0}^{\infty} P_n^1 (\cos \theta) \, A_{on} \frac{1}{k_1 r} [k_1 r j_n (k_1 r)]' \, e^{-j\omega t} + E_{\theta_0} \tag{10}$$

$$H_{\phi}^{i} = \frac{k_1}{j\omega\mu_1} \sum_{n=0}^{\infty} A_{on} P_n^1 (\cos \theta) j_n (k_1 r) \, e^{-j\omega t} \tag{11}$$

and the field components E_r^e, E_θ^e and H_ϕ^e outside the sphere are obtained by replacing the spherical Bessel function $j_n (k_1 r)$ by the spherical Hankel function $h_n^{(1)} (k_0 r)$ and A_{on} by B_{on} in the expressions for E_r^i, E_θ^i and H_ϕ^i respectively, where

$$k_0 = \omega \, [\mu_0 \, (\epsilon_0 + j\sigma_0/\omega)]^{1/2} \quad \text{and} \quad k_1 = \omega \, [\mu_1 \, (\epsilon_1 + j\sigma_1/\omega)]^{1/2}.$$

Applying the boundary condition that at $r = a$, $E_\theta^e = E_\theta^i$ and $H_\phi^e = H_\phi^i$, we obtain

$$B_{on} = \frac{C_{on} (a) \, j\omega\mu_0}{k_1 k_0 \, h_n^{(1)} (k_0 a)} \, \frac{1}{Z_n} \tag{12}$$

$$A_{on} = \frac{\mu_1}{\mu_0} \, \frac{k_0 \, h_n^{(1)} (k_0 a)}{k_1 \, j_n (k_1 a)} \, B_{on} \tag{13}$$

where

$$Z_n = \frac{j\omega\mu_1}{k_1^2 \, a} \, \frac{[k_1 a \, j_n (k_1 a)]'}{j_n (k_1 a)} - \frac{j\omega\mu_0}{k_0^2 \, a} \cdot \frac{[k_0 a \, h_n^{(1)}(k_0 a)]'}{h_n^{(1)} (k_0 a)} \tag{14}$$

Z_n is the difference between the radial wave impedances E_θ^i/H_ϕ^i inside and E_θ^e/H_ϕ^e outside the sphere. Free oscillation of the sphere results when these impedances are matched, that is, when $Z_n = 0$, an equation whose roots determine the resonant frequencies of the natural modes of oscillation.

The field is thus determined uniquely both inside and outside the dielectric sphere for each value of n. As explained earlier, this shows that TM_{on} modes exist for $n = 1$, 2, 3, ... for the dielectric sphere. There are also an infinite number of resonant frequencies for each TM_{on} mode, and the fields become very large at these resonant frequencies.

For symmetric TE modes, the excitation is $H_0 e^{-j\omega t}$ applied in the z direction over an annular ring of width $\triangle s \to 0$ in the plane $z = z_1 = a \cos \theta_1$. The field components of symmetric TE modes are H_r, H_θ and E_ϕ. Proceeding in a similar manner for the TM modes, it can be shown that the resonant frequencies for TE modes are given by the equation

$$\frac{[k_1 a j_n (k_1 a)]'}{j_n (k_1 a)} = \frac{\mu_1}{\mu_0} \frac{[k_0 a \, h_n^{(1)} \, (k_0 a)]'}{h_n^{(1)} (k_0 a)} \tag{15}$$

3. Calculation of the resonant frequencies and amplitudes of the field components

Equations (14) and (15) have been solved by using iteration method with the aid of an IBM 360 digital computer, and the roots have been determined for several TE and TM modes for dielectric spheres embedded in free space ($\sigma_0 = 0$). These roots are complex of the form $\omega_r = \omega_{r_1} + j\omega_{r_2}$ where $\omega_{r_1} = 2\pi f_{r_1}$, gives the free oscillation radian frequency, and $1/\omega_{r_2} = 1/2\pi f_{r_2}$ gives the relaxation time.

Figs. 2 (a) and 2 (b) show the resonant frequencies f_{r_1} and f_{r_2} for different TM modes as functions of (a/v_d), where

$$v_d = \frac{1}{\sqrt{\mu_1 \epsilon_1}} = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

is the intrinsic phase velocity in the low-loss dielectric sphere (loss tangent $= 0 \cdot 005$), where $\mu_r = \mu_1/\mu_0$ and $\epsilon_r = \epsilon_1/\epsilon_0$. Figs. 2 (c) and 2 (d) show the resonant frequencies f_{r_1} and f_{r_2} for different TE modes as functions of (a/v_d). Figs. 3 (a) and 3 (b) show the resonant frequencies f_{r_1} and f_{r_2} respectively for the different TE and TM modes as functions of (a/λ_r), λ_r being the wavelength in the dielectric at resonance. ϵ_r has been taken as $2 \cdot 56$ in this case. For natural oscillation of a resonator the Q-factor can be defined in terms of the real part ω_{r_1} and imaginary part ω_{r_2} of the complex angular frequency ω_{r_2} of resonance (see Appendix) as

$$Q = \frac{\omega_{r_1}}{2\omega_{r_2}} = \frac{f_{r_1}}{2f_{r_2}} \tag{16}$$

where f_{r_1} and f_{r_2} denote the real and imaginary parts of the resonant frequency $f_r (\omega_r = 2\pi f_r)$ respectively. Since Q is determined by the ratio (eqn. 16) and f_{r_1} and f_{r_2} are independent of a/λ_r [see Figs. 3 (a) and 3 (b)], for any TE_{on} and TM_{on} mode it may, therefore, be concluded that the Q-factor is independent of a/λ_r for the above modes. Table I gives the Q-factors for several TE and TM modes for the first and second roots of eqns. (14) and (15).

Plots of $a \mid A_{on} \mid$ and $a \mid B_{on} \mid$ as functions of λ_0/a with relative permittivity ϵ_r and permeability μ_r as parameters for TM_{on} modes ($n = 1$ to 6) with source of excitation applied across different planes indicated by θ_1 show that (Figs. 4–8)

  (i) the peak values differ in magnitude for TM_{01} and TM_{02} modes and for different values of ϵ_r and μ_r.

 (ii) the peak values of $a \mid A_{on} \mid$ are greater than the peak values of $a \mid B_{on} \mid$.

(iii) the position of the peaks along the a/λ_0 axis shift depending on the parameters;

 (iv) the plane of excitation has very little effect on the position of peaks though it influences the magnitude of the peak. A knowledge of the position of minima for the amplitude constant $\mid A_{on} \mid$ can be utilised to select proper exciting wave-
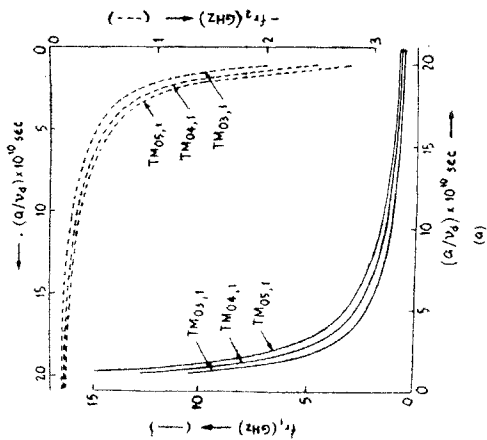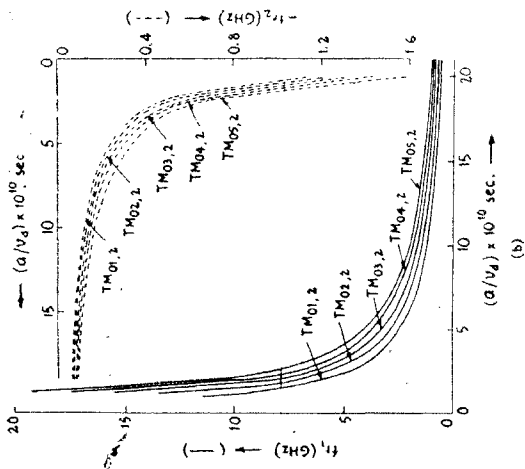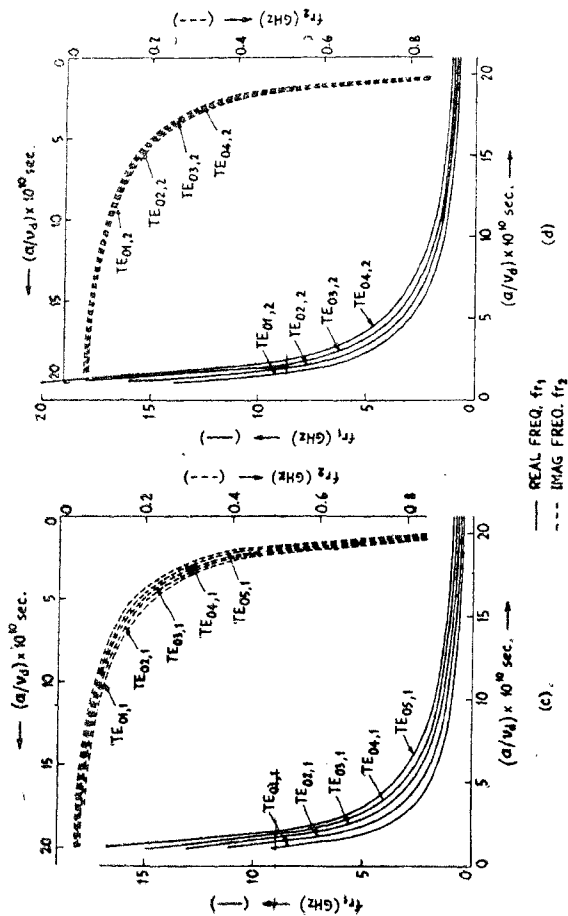
Fig. 2. Resonant frequencies f_{r_1} and f_{r_2} as functions of (a/v_d) for TM and TE modes. $\epsilon_r = 2.56$.

length λ_0 so that the absorption of power by biological spheres can be kept to a minimum and thus reduce the temperature rise and consequently minimise the risk of biological damage.

Table I

Q-factor for TE and TM modes; $\epsilon_r = 2 \cdot 56$

Mode	Q	
	I Root	II Root
TE_{01}	4·4078	6·6597
TE_{02}	5·6315	7·8558
TE_{03}	6·9802	9·1110
TE_{04}	8·4959	10·4409
TE_{05}	10·2285	. .
TM_{01}	. .	5·1194
TM_{02}	. .	5·6167
TM_{03}	2·6415	5·9263
TM_{04}	2·5969	6·0710
TM_{05}	2·7734	6·0579

The plot of the real and imaginary parts of $a \mid B_{0n} \mid$ (Fig. 9) show that both $Re \, (a \, B_{0n})$ and $I_m \, (a \, B_{0n})$ became zero and hence the amplitude constant for the field outside the sphere becomes zero at a particular value of a/λ_0 irrespective of the value of θ_2. This may lead to the concept of confined mode for a particular value of a/λ_0.
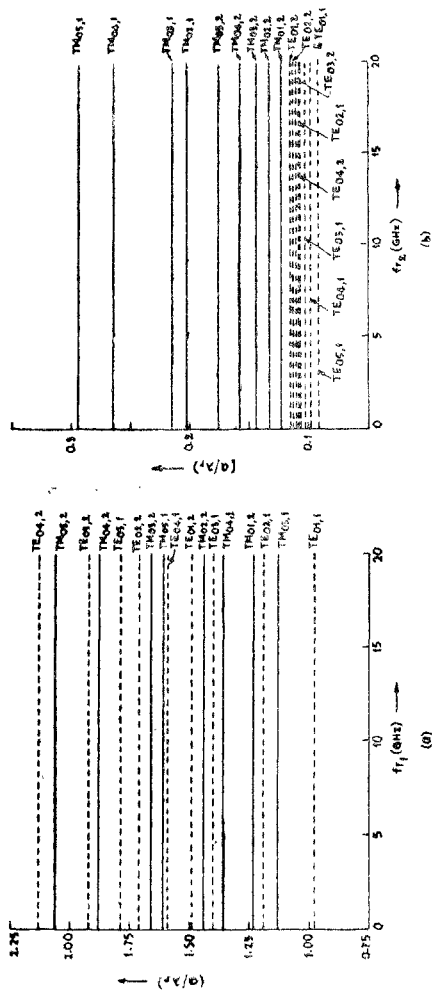
4. Resonant frequencies of biological spheres

Biological tissues like muscle, brain matter, etc., may be approximately considered as homogeneous lossy dielectrics[11-13]. Using the expressions[12]

$$\frac{\epsilon_1}{\epsilon_0} = \epsilon_r = 5 \left[12 + \frac{f}{f_0} \right]^2 \bigg/ \left[1 + \left(\frac{f}{f_0} \right)^2 \right] \tag{17}$$

and

$$\sigma_1 = 6 \left[1 + 62 \left(\frac{f}{f_0} \right)^2 \right] \bigg/ \left[1 + \left(\frac{f}{f_0} \right)^2 \right] \text{ mmhos/cm} \tag{18}$$

FIG. 3.   Resonant frequencies f_{r_1} and f_{r_2} as functions of (a/λ_r) for TM and TE modes. $\epsilon_r = 2.56$.
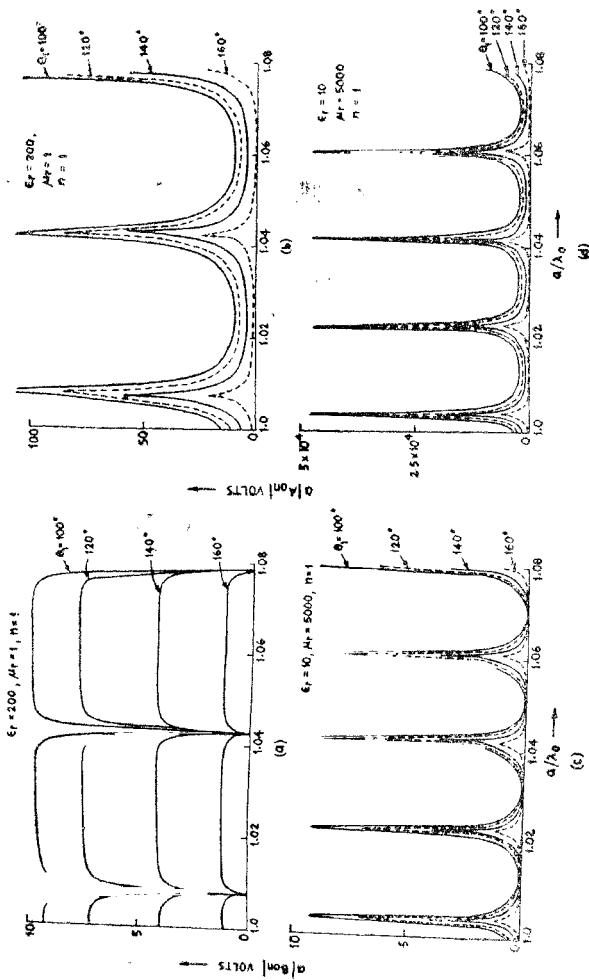
FIG. 4. $|aB_{on}|$ and $|aA_{on}|$ as functions of (a/λ_0) for $n=1$ and (i) $\epsilon_r = 200$, $\mu_r = 1$, and (ii) $\epsilon_r = 10$, $\mu_r = 5000$, for different angles θ_1.
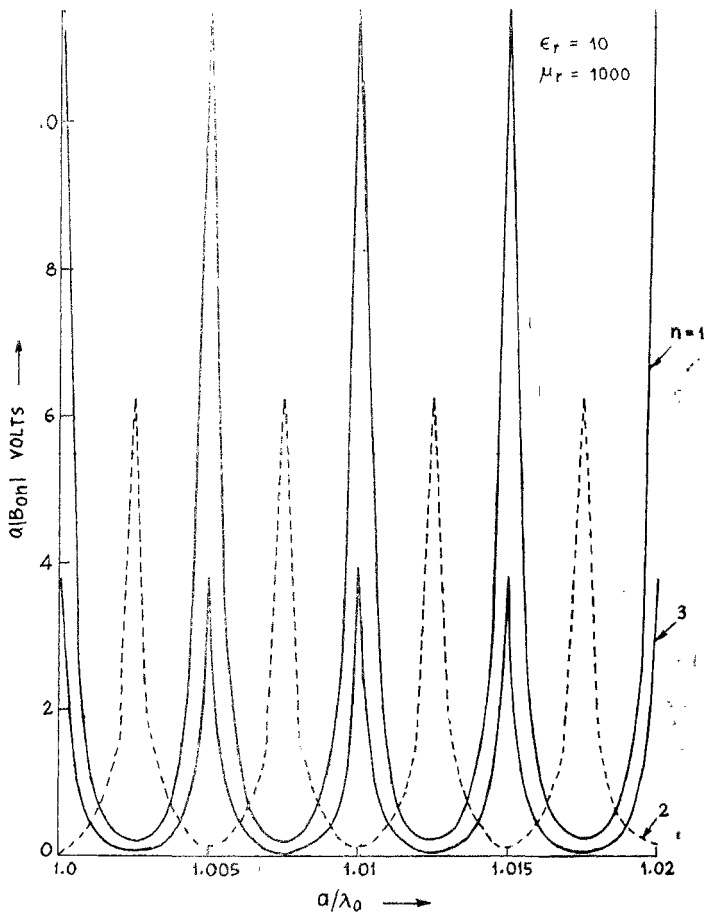
FIG. 5. $|aB_{on}|$ as function of (a/λ_0) for $n = 1, 2, 3$, $\theta_1 = 130°$, and $\epsilon_r = 10$, $\mu_r' = 1000$,
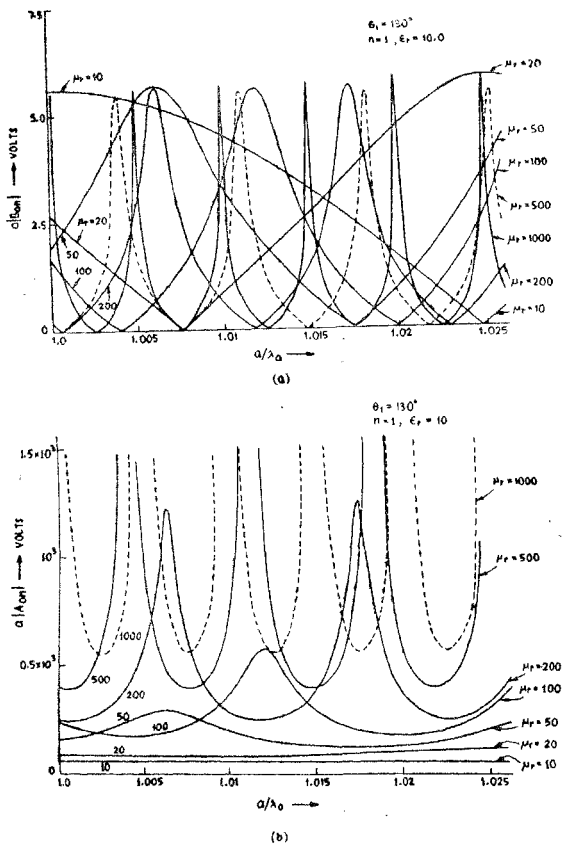
(a)

(b)

FIG. 6. $|aB_{on}|$ and $|aA_{na}|$ as functions of a/λ, for $\theta_1 = 130°$, $n = 1$, $\epsilon_r = 10$, for different values of μ_r.
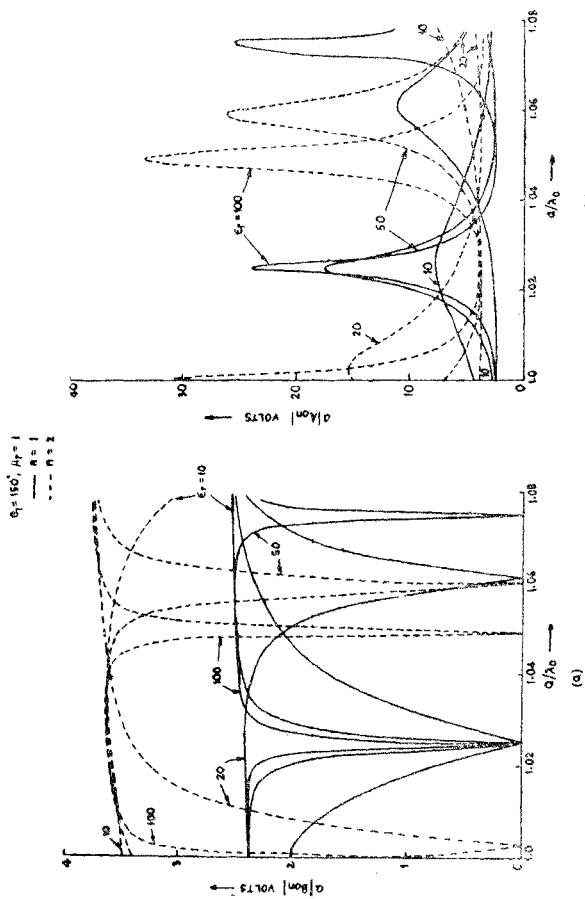
FIG. 7. $|aB_{0n}|$ and $|aA_{0n}|$ as functions of a/λ_0 for $\theta_i = 150°$, $\mu_r = 1$, $n = 1, 2$ and for different values of ϵ_r.
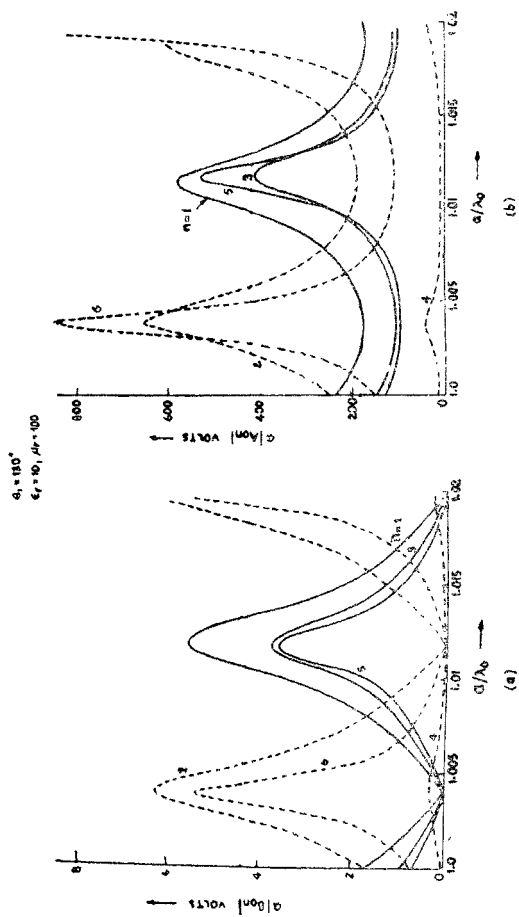
FIG. 8. $|aB_{0n}|$ and $|aA_{0n}|$ as functions of a/λ_0 for $\theta_1 = 130^\circ$, $\epsilon_r = 10$, $\mu_r = 100$ for $n = 1, 2, 3, 4, 5, 6$.
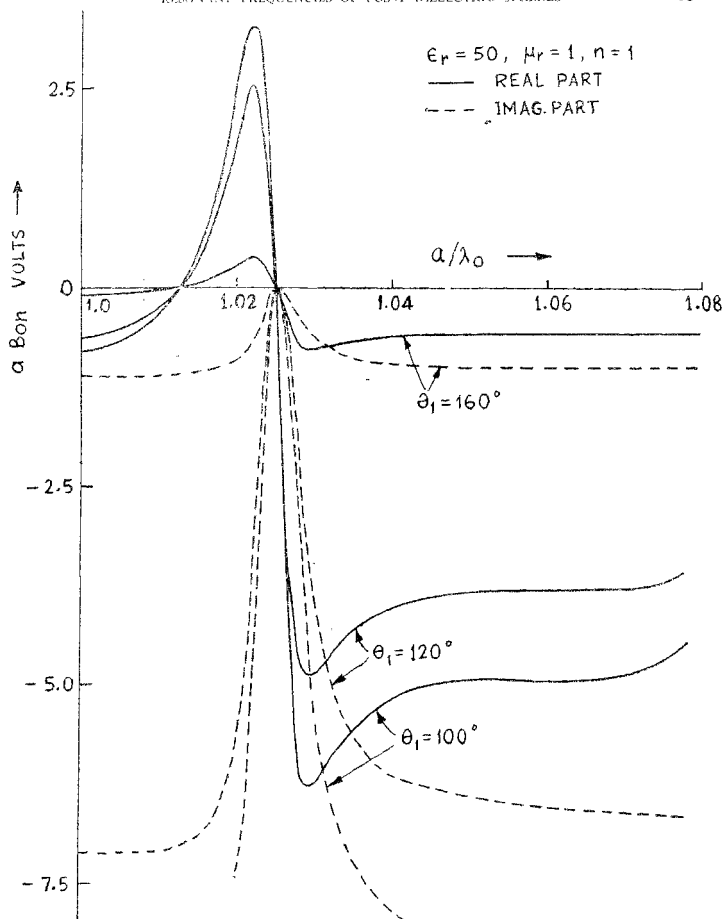
FIG. 9. Real and imaginary parts of (aB_{on}) as functions of a/λ_0 for $n = 1$, $\epsilon_r = 50$, $\mu_r = 1$ for different angles θ_1.
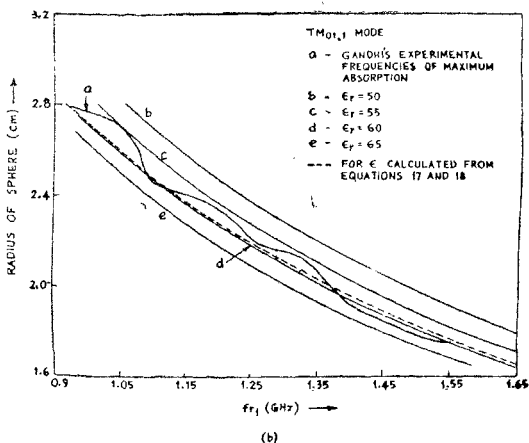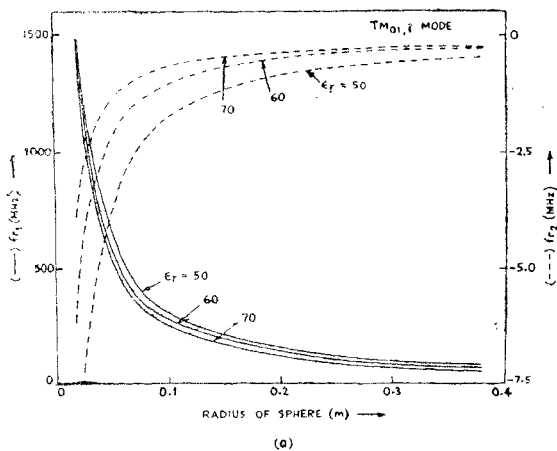
(a)

(b)

FIG. 10 (a). Real and imaginary resonant frequencies f_{r_1} and f_{r_2} of biological spheres. (b). Real resonant frequency f_{r_1} for the TM_{011} mode of biological spheres as compared with experimental results[14].

for brain matter, where $f_0 = 20 \times 10^9$ Hz, the resonant frequencies of biological spheres of brain matter of varying radius ' a ' have been calculated. The real and imaginary parts of f_{r_1} and f_{r_2} are shown in Fig. 10 (a). Fig. 10 (b) shows the real resonant frequency f_{r_1} for the TM_{011} mode as a function of the radius ' a ', and is compared with experimental results obtained by previous workers[14].

The theoretical analysis of frequency of maximum absorption shows a monotonic decrease with radius of the sphere having any value of ϵ, whereas the experimental curve reported by Gandhi[14] shows an oscillatory nature. However, the analysis for the sphere with values of ϵ_r and σ_1 calculated from eqns. (17) and (18) show fair agreement between the theoretical and experimental values.

5. Conclusions

The following conclusions can be drawn from the above study of the resonant properties of dielectric spheres :

   (i) The dielectric sphere can resonate both in TM_{on} and TE_{on} modes, and the resonant frequencies are complex of the form $\omega_r = \omega_{r_1} + j\omega_{r_2}$, ω_{r_1} being the real resonant frequency and $1/\omega_{r_2}$ being the relaxation time.

  (ii) At a particular frequency (or wavelength) for a sphere of given radius, several TE or TM modes may exist as shown in Figs. 2 and 3.

 (iii) As shown by Figs. 4, 6, 7 and 8 the field inside the sphere at resonance is very much stronger than the field outside the sphere.
       (The above conclusions, though arrived at by previous workers also, have been written for the sake of completeness.)

 (iv) The fields inside and outside the sphere for high values of ϵ_r behave quite differently from the fields for high values of μ_r as shown in Figs. 4 to 8.

  (v) The calculated resonant frequencies for the TM_{011} mode (first root of TM_{01} mode) of biological spheres, having the properties of brain matter, agree fairly well with the frequencies of maximum absorption of electromagnetic radiation as reported by experimental workers, as shown in Fig. 10 (b).

 (vi) Other workers[15] have reported that there is a region of maximum heating in the frequency *versus* radius diagram for a lossy sphere having the same electrical characteristics as brain tissue. This can also be seen from Fig. 2, where the resonant frequencies of various TM and TE modes occur within a certain region for a sphere of given radius and given electrical properties.

(vii) As the Re (aB_{on}) and I_m (aB_{on}) become zero at a particular value of a/λ_0, it may be said that the modal oscillation may be completely confined inside the sphere at this value of a/λ_0 indicating possibly the absence of radiation.

I.I.Sc.—4

6. Acknowledgements

References

1. DEBYE, P.   Der Lichtdruck auf Kugeln von Behebigen Material, *Ann. Phys.*, 1909, **30**, Ser. 4, 57–136.

2. RICHTMEYER, R. D.   Dielectric resonators, *J. Appl. Phys.*, 1939, **10** (6), 391.

3. OKAYA, A.   The rutile microwave resonators, *Proc. I.R.E.* (Correspondence), 1960, **48**, 1921.

4. OKAYA, A. AND   The dielectric microwave resonator, *Proc. I.R.E.*, 1962, **50**, BARASH, L. F.   2081–2092.

5. GASTINE, M., COURTOIS, L.   Electromagnetic resonances of free dielectric spheres, *IEEE Trans.* AND DORMANN, J. L.   *M.T.T.*, 1967, **MTT-15** (12), 694–700.

6. SAGER, O. AND TISI, F.   On eigen modes and forced resonance modes of dielectric spheres, *Proc. I.E.E.E.* (Letters), 1968, **56**, 1593–1594.

7. AFFOLTER, P. AND   Electromagnetic resonances and *Q* factors of lossy dielectric ELIASSON, B.   spheres, *I.E.E.E. Trans.*, 1973, **MTT-21** (9), 573–578.

8. VAN BLADEL, J.   On the resonances of a dielectric resonator of very high permittivity, *I.E.E.E. Trans.*, 1975, **MTT-23** (2), 199–208.

9. VAN BLADEL, J.   The excitation of dielectric resonators of very high permittivity, *I.E.E.E. Trans.*, 1975, **MTT-23** (2), 208–217.

10. CHATTERJEE, R.   Electromagnetic boundary value problem of the dielectric sphere excited by delta-function electric and magnetic sources, *Jour. Ind. Inst. Sci.*, 1973, **55** (1), 16–31.

11. JOHNSON, C. C. AND   Nonionizing electromagnetic wave effects in biological materials GUY, A. W.   and systems, *Proc. I.E.E.E.*, 1972, **60** (6), 692–718.

12. JONES, W. T. AND   Resonance absorption of microwaves by the human skull, SPIEGEL, R. J.   *I.E.E.E. Trans. Biomed. Engr. Communications*, 1974, **BME-21**, 46-48.

13. SCHWANN, H. P.   Survey of microwave absorption characteristics of body tissues, *Proc. 2nd Tri-Service Conf. on Biological Effects of Microwave Energy*, 1968, pp. 126–145.

14. GANDHI, O. P.   Orientational and frequency effects on whole animal absorption of R.F. energy, *Jour. Ind. Inst. Sci.*, 1974, **56** (7), 315–364.

15. KRITIKOS, H. N. AND   The distribution of heating potential inside lossy spheres, *I.E.E.E.* SCHWANN, H. P.   *Trans. Biomedical Engineering*, 1975, **BME-22** (6), 457–463.

16. CURTOIS, L.   A new low level limiter for centimeter and millimeter waves, *J. Appl. Phys.*, 1967, **38**, 1415–1476.

Appendix

Derivation of the Q-factor of the sphere

The transfer impedance $Z(\omega)$ of a cavity resonator is defined by

$$e = iZ(\omega) \tag{A.1}$$

where 'e' is the e.m.f. induced in the output loop and 'i' is the current of constant amplitude, but of variable frequency f, maintained in the input loop by a suitable source.

For a 'high-Q' cavity, $Z(\omega)$ in the vicinity of a resonant angular frequency ω_r is approximately given as

$$Z(\omega) = \frac{K}{1 + j2Q\,\dfrac{\delta\omega_r}{\omega_r}} \tag{A.2}$$

where

$$\delta\omega_r = \omega - \omega_r.$$

The quantity K depends upon the size and orientation of the coupling loops, while Q and ω_r are independent of these factors provided that the coupling is weak.

Eqn. (A.1) expresses the relation between input and output amplitudes under the conditions of forced oscillations, with the factor $e^{j\omega t}$ representing undamped sinusoidal variation in time.

If the input is removed now, the natural oscillations still persist, but with an exponential decay with time. The frequency of oscillations is the same as the resonant frequency of the cavity within the degree of approximation imposed by (A.2). A suitable time function is of the form

$$e^{(-\omega_{r_2} + j\omega_{r_1})\,t} \qquad \text{or} \qquad e^{j\,(\omega_{r_1} + j\omega_{r_2})\,t}$$

in which ω_{r_2} is a real positive constant, known as the decay factor. When natural oscillations are under consideration, ω_r must be replaced by $\omega_{r_1} + j\omega_{r_2}$ in (A.2) so that $\delta\omega_r$ becomes $j\omega_{r_2}$.

When the input current $i = 0$, the output, e, can have a finite value only if $Z(\omega) = \infty$,

.e.,

$$\frac{1}{Z(\omega_{r_1} + j\omega_{r_2})} = 0$$

i.e.,

$$1 + j \frac{2Q(j\omega_{r_2})}{\omega_{r_1}} = 0$$

or

$$Q = \frac{\omega_{r_1}}{2\omega_{r_2}}. \tag{A.3}$$