Nonnegative solution of linear equations

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Abstract

We give a method to obtain a nonnegative solution of any system of linear equations, if such a solution exists. The method writes linear equations as a linear programming problem and then solves this problem using a Simplex method.

Key words: Artificial basis technique, linear programming, nonnegative solution, Simplex method.

1. Introduction

In many physical problems, the negative of quantities like path, matter, time, etc., does not arise. Any such problem giving rise to linear equations involving such unknown quantities needs nonnegative solution.

The method described here investigates equations $By = g$, consistent or not, underdetermined or overdetermined, as a linear programming (*lp*) problem and gives a nonnegative solution y when it exists. To solve the *lp* problem the method involves a particular form of the artificial basis technique[1, 2].

2. Definitions

Extended (Simplex) tableau

Consider the *lp* problem

$$\text{Minimize } f = c^t x \text{ subject to } Ax = b, \quad x \geqslant 0. \tag{1}$$

The initial extended tableau (*i.e.*, Extended tableau 0) for this *lp* problem,

where

$$d_j = c_{n+1}a_{1j} + c_{n+2}a_{2j} + \cdots + c_{n+m}a_{mj} - c_j \quad j = 1\,(1)\,n$$

$$= c_{n+j} \times 1 - c_{n+j} = 0 \quad j = n+1\,(1)\,n+m$$

$$d_{n+m+1} = c_{n+1}b_1 + c_{n+2}b_2 + \cdots + c_{n+m}b_m.$$

is

	(c_1)		(c_{j_0})		(c_n)	(c_{n+1})		(c_{n+i_0})		(c_{n+m})		
	x_1	$\cdot\cdot$	x_{j_0}	$\cdot\cdot$	x_n	x_{n+1}	$\cdot\cdot$	x_{n+i_0}	$\cdot\cdot$	x_{n+m}	b	
$(c_{n+1})\ x_{n+1}$	a_{11}		a_{1j_0}		a_{1n}	1		0		0	b_1	
	$\cdot$											
	$\cdot$											
$(c_{n+i_0})\ x_{n+i_0}$	a_{i_01}		$a_{i_0j_0}$		a_{i_0n}	0		1		0	b_{i_0}	(2)
	$\cdot$											
	$\cdot$											
$(c_{n+m})\ x_{n+m}$	a_{m1}		a_{mj_0}		a_{mn}	0		0		1	b_m	
	d_1		d_{j_0}		d_n	d_{n+1}		d_{n+i_0}		d_{n+m}	d_{n+m+1}	

Objective function

The function $f = f(x) = c^t x$, or equivalently, $f(x) = c_1x_1 + \cdots + c_{n+m}x_{n+m}$ which is to be optimized (minimized or maximized) is called the objective function.

The Checking rule for a Simplex tableau

The foregoing relationship between d_j and c_j, a_{ij} holds in all tableaux. This relationship is referred to as the *Checking rule* for a tableau. Satisfaction of this rule is necessary for a tableau to be correct but it is not sufficient (*i.e.*, the rule may be satisfied even if a computational mistake occurs).

Note : The role of $c_1, \ldots, c_{n+m}$ is over as soon as $d_1, \ldots, d_{n+m}$ are computed. The subsequent extended tableaux (*viz.*, Extended tableaux 1, 2, ..) are computed from the Extended tableau 0 using Simplex rules. The d_j-row has to be nonpositive (for the minimization problem considered here) in the optimal (final) Extended tableau.

Restricted (Simplex) tableau

Consider the *lp* problem (1). The initial restricted tableau (*i.e.*, Restricted tableau 0) for this *lp* problem is the Extended tableau 0 with columns containing unit vectors deleted. The subsequent restricted tableaux are computed from the Restricted tableau 0 using Simplex rules[5]. The d_j-row has to be nonpositive (for the minimization problem considered here) in the optimal (final) Restricted tableau.

3. The problem

$$\left.\begin{array}{l}\text{Obtain a nonnegative solution of } By = g \text{ (if it exists)}\\ \text{where } B = (b_{ij}) \text{ is a given } m \times n \text{ matrix, } g = (g_i) \text{ is a given}\\ \text{nonnegative } m\text{-vector, and } y = (y_j) \text{ is an } n\text{-vector.}\end{array}\right\} \quad (3)$$

Note: There is no loss of generality in considering $g \geqslant 0$. If this is not so, then multiply the equations with negative g_i by -1.

4. Existence of a nonnegative solution

$By = g$ has a nonnegative solution y if and only if $B^t z \geqslant 0$, $g^t z < 0$ has no solution z. Equivalently, $By = g$ has no nonnegative solution y if and only if $B^t z \geqslant 0$, $g^t z < 0$ has a solution z.

For proof see Farkas[3] and Vajda[4].

This result is not of immediate use. However, the method tells if a nonnegative solution of $By = g$ does not exist. In fact, the necessary and sufficient condition for $By = g$ to have a nonnegative solution is the method producing one.

5. The method

Write (3) as an *lp* problem and solve this problem using an artificial basis technique, as below :

(i) *Equivalent lp problem*

Let y and B be now extended to $(n+m)$-vector x and $m \times (n+m)$ matrix A, respectively. Further, let the last m columns of A form an $m \times m$ unit matrix I_m. The *lp* problem equivalent to (3) is

$$\left.\begin{array}{l} \text{Obtain } x \text{ so that} \\ \text{Min } f = x_{n+1} + \cdots + x_{n+m} = 0 : \text{ Objective function} \\ \text{subject to} \\ Ax = b : \text{Constraints} \\ x \geqslant 0 : \text{ Nonnegativity conditions} \end{array}\right\} \quad (4)$$

(ii) *Artificial basis technique in 'Extended tableau'*

Step 1: Set up the extended Simplex tableau for (4), and write the coefficients (in parentheses) which x_j have in the objective function and the last row, *i.e.*, d_j-row using the Checking rule as below :

	(0)		(0)		(0)	(1)		(1)		(1)	
	x_1	$\cdot\cdot$	x_{j_0}	$\cdot\cdot$	x_n	x_{n+1}	$\cdot\cdot$	x_{n+i_0}	$\cdot\cdot$	x_{n+m}	b
(1) x_{n+1}	a_{11}		a_{1j_0}		a_{1n}	1		0		0	b_1
	$\vdots$										

$$\begin{array}{lccccccccl}
(1)\ x_{n+i_0} & a_{i_0 1} & a_{i_0 j_0} & a_{i_0 n} & 0 & 1 & 0 & b_{i_0} & (5) \\
 & \vdots & & & & & & & \\
(1)\ x_{n+m} & a_{m1} & a_{mj_0} & a_{mn} & 0 & 0 & 1 & b_m & \\
 & d_1 & d_{j_0} & d_n & 0 & 0 & 0 & d_{n+m+1} &
\end{array}$$

Step 2 (pivot selection): Let d_{j_0} be positive. Consider then, for all positive a_{ij_0}, the ratios b_i/a_{ij_0} and take the smallest. If this is obtained for i_0 then call $p = a_{i_0 j_0}$ the pivot (marked with a plus). Go to Step 3. Otherwise (*i.e.*, if there exists no d_{j_0} which is positive) the present tableau is final and it either indicates no solution of $By = g$ or gives a solution.

Step 3 (next-tableau computation): Replacing x_{n+i_0} by x_{j_0} obtain the next tableau as follows:

$$\left.\begin{array}{lcccccccccc}
 & x_1 & \cdots & x_{j_0} & \cdots & x_n & x_{n+1} & \cdots & x_{n+i_0} & \cdots\ x_{n+m} & b \\
x_{n+1} & & & 0 = a_{1j_0} - p \cdot a_{1j_0}/p & & & & & -a_{1j_0} & & \\
 & & & \vdots & & & & & & & \\
x_{j_0} & a_{i_0 1}/p & & 1 & & a_{i_0 n}/p & 0 & & 1/p & 0 & b_{i_0}/p \\
 & & & \vdots & & & & & & & \\
x_{n+m} & & & 0 = a_{mj_0} - p \cdot a_{mj_0}/p & & & & & -a_{mj_0}/p & & \\
 & & & 0 & & & & & -d_{j_0}/p & &
\end{array}\right\} \quad (6)$$

The blank positions are filled in as follows:

$$\begin{aligned}
a_{ij} &\leftarrow a_{ij} - a_{ij_0} a_{i_0 j}/p \\
d_j &\leftarrow d_j - d_{j_0} a_{i_0 j}/p \\
b_i &\leftarrow b_i - a_{ij_0} b_{i_0}/p.
\end{aligned} \quad (7)$$

All the entries on the right hand side of (7) are the elements of the previous tableau.

Both (6) and (7) may be precisely written as $(p = a_{i_0 j_0})$

$$\begin{aligned}
&\textit{pivot row} \leftarrow \text{pivot row}/p \\
&\text{(any other) } i\text{-th row} \leftarrow i\text{-th row} - a_{ij_0} \times \textit{pivot row}
\end{aligned} \quad (8)$$

Note : Pivot row is the row containing the pivot. Pivot column is the column containing the pivot.

Step 4 (termination condition): If the bottom row, *i.e.*, d_j-row excluding the last element is nonpositive, and if none of $x_{n+1}, \cdots, x_{n+m}$ occurs in the basis with a nonzero value then the solution is reached. Otherwise go to Step 2.

(iia) *Artificial basis technique in 'restricted tableau'*

Step 1 *a*: Set up the restricted Simplex tableau for (4), and write the coefficients (in parentheses) which x_j have in the objective function and the last row, *i.e.*, d_j-row using the Checking rule as below.

	(0)		(0)		(0)		
	x_1	$\cdot\cdot$	x_{j_0}	$\cdot\cdot$	x_n	b	
(1) x_{n+1}	a_{11}		a_{1j_0}		a_{1n}	b_1	
	$\cdot$						
	$\cdot$						
(1) x_{n+i_0}	a_{i_01}		$a_{i_0j_0}$		a_{i_0n}	b_{i_0}	(5 *a*)
	$\cdot$						
	$\cdot$						
(1) x_{n+m}	a_{m1}		a_{mj_0}		a_{mn}	b_m	
	d_1		d_{j_0}		d_n	d_{n+1}	

Note: Here d_{n+1} corresponds to d_{n+m+1} of the extended Simplex tableau (2).

Step 2 *a* (pivot selection): It is the same as Step 2 with replacement of tableau (5) by tableau (5 *a*).

Step 3 *a* (next-tableau computation): Having interchanged x_{j_0} and x_{n+i_0} obtain the next tableau as follows:

	x_1	$\cdot\cdot$	x_{n+i_0}	$\cdot\cdot$	x_n	b	
x_{n+1}			$-a_{1j_0}/p$				
			$\cdot$				
			$\cdot$				
x_{j_0}	a_{i_01}/p		$1/p$		a_{i_0n}/p	b_{i_0}/p	(6 *a*)
			$\cdot$				
			$\cdot$				
x_{n+m}			$-a_{mj_0}/p$				
			$-d_{j_0}/p$				

The blank positions are filled in as follows:

$$a_{ij} \leftarrow a_{ij} - a_{i_0 j} a_{i j_0} / p$$

$$d_j \leftarrow d_j - a_{i_0 j} d_{j_0} / p \quad (7\,a)$$

Note :

- The foregoing two 'replacements' are actually identical when we consider the last row (*i.e.*, d_j-row) as just another row like the rows of (a_{ij}).
- The right hand side elements are the elements of the foregoing tableau throughout the computation.

Step 4 *a* (termination condition): It is the same as Step 4.

6. Proof of the method

The method is a particular case of the *M*-method[1, 2]. So, all the properties and inferences regarding the *M*-method hold good here also. However, all the tableaux in solving the problem are equivalent in the sense that the solution or solutions of the original equations remain invariant throughout. If there exists a nonnegative solution then the final tableau will give it. On the contrary, if there is none then one or more artificial variables will be in the basis (in the final tableau) with a nonzero value.

7. Examples

(i) *Nondegenerate case* (*i.e.*, rank of coefficient matrix = number of equations = 2): Obtain a nonnegative solution of

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ -4 & -2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The equivalent *lp* problem is: Compute $x = (x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7)^t$ so that Min $f = x_6 + x_7 = 0$ subject to $Ax = b$, $x \geqslant 0$ where

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 1 & 0 \\ -4 & -2 & 3 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We now write both extended and restricted tableaux to obtain the solution and to show how the restricted tableau differs from the extended one.

Extended tableau 0

	(0) x_1	(0) x_2	(0) x_3	(0) x_4	(0) x_5	(1) x_6	(1) x_7	b
(1) x_6	1	2	1	1	0	1	0	1
(1) x_7	-4	-2	3^+	0	1	0	1	2
	-3	0	4	1	1	0	0	3

Restricted tableau 0

	(0) x_1	(0) x_2	(0) x_3	(0) x_4	(0) x_5	b
(1) x_6	1	2	1	1	0	1
(1) x_7	-4	-2	3^+	0	1	2
	-3	0	4	1	1	3

Extended tableau 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
x_6	$\frac{7}{3}^+$	$\frac{8}{3}$	0	1	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{1}{3}$
x_3	$-\frac{4}{3}$	$-\frac{2}{3}$	1	0	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$
	$\frac{7}{3}$	$\frac{8}{3}$	0	1	$-\frac{1}{3}$	0	$-\frac{4}{3}$	$\frac{1}{3}$

Restricted tableau 1

	x_1	x_2	x_7	x_4	x_5	b
x_6	$\frac{7}{3}^+$	$\frac{8}{3}$	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{1}{3}$
x_3	$-\frac{4}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$
	$\frac{7}{3}$	$\frac{8}{3}$	$-\frac{4}{3}$	1	$-\frac{1}{3}$	$\frac{1}{3}$

Extended tableau 2

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
x_1	1	$\frac{8}{7}$	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$
x_3	0	$\frac{6}{7}$	1	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{6}{7}$
	0	0	0	0	0	-1	-1	0

Restricted tableau 2

	x_6	x_2	x_7	x_4	x_5	b
x_1	$\frac{3}{7}$	$\frac{8}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$
x_3	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{6}{7}$
	-1	0	-1	0	0	0

Hence a nonnegative solution is $x = [\frac{1}{7}\ 0\ \frac{6}{7}\ 0\ 0]^t$.

(ii) *Degenerate case* (redundant equations): Obtain a nonnegative solution of

$$\begin{bmatrix} -1 & 2 & 3 & 3 \\ 2 & 5 & 6 & 3 \\ -5 & -8 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ -25 \end{bmatrix}$$

Setting up the equivalent *lp* problem the restricted tableaux are

Restricted tableau 0

	(0) x_1	(0) x_2	(0) x_3	(0) x_4	b
(1) x_5	-1	2	3^+	3	7
(1) x_6	2	5	6	3	16
(1) x_7	5	8	9	3	25
	6	15	18	9	48

Restricted tableau 1

	x_1	x_2	x_5	x_4	b
x_3	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{7}{3}$
x_6	4	1^+	-2	-3	2
x_7	8	2	-3	-6	4
	0	3	-6	-9	6

Note: The last equation has been multiplied by -1 to make b_3 positive (refer *Restricted tableau* 0).

Restricted tableau 2

	x_1	x_6	x_5	x_4	b
x_3	-3	$-\frac{2}{3}$	$\frac{5}{3}$	3	1
x_2	4	1	-2	-3	2
x_7	0	-2	1	0	0
	-12	-3	0	0	0

The artificial variable x_7 remains in the basis with a zero value. A nonnegative solution is $x = [0 \quad 2 \quad 1 \quad 0]^t$.

(iii) *Inconsistent equations*: Obtain a nonnegative solution (if any) of

$$\begin{bmatrix} 5 & 3 & 2 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}$$

Setting up the equivalent *lp* problem we write the restricted tableaux as below :

Restricted tableau 0

	(0)	(0)	(0)	
	x_1	x_2	x_3	b
(1) x_4	5	3	2	10
(1) x_5	2	1	2	5
(1) x_6	4	2	4	1
	11	6	8	16

Restricted tableau 1

	x_1	x_6	x_3	b
x_4	-1	$-\frac{3}{2}$	-4	$\frac{17}{2}$
x_5	0	1	0	$\frac{9}{2}$
x_2	2	$\frac{1}{2}$	2	$\frac{1}{2}$
	-1	-3	-4	13

The last row except the last element (*viz.*, 13) is nonpositive and two artificial variables, *viz.*, x_4 and x_5 are still in the basis with nonzero values. Hence the equations have no nonnegative solution. In fact, the equations have no solution at all.

(iv) *Solution with one negative element*: Obtain a nonnegative solution (if any) of

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Restricted tableau 0

	(0)	(0)	(0)	
	x_1	x_2	x_3	b
(1) x_4	2	1	3^+	2
(1) x_5	1	1	1	1
(1) x_6	1	2	1	2
	4	4	5	5

Restricted tableau 1

	x_1	x_2	x_4	b
x_3	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
x_5	$\frac{1}{3}^+$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
x_6	$\frac{1}{3}$	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$
	$\frac{2}{3}$	$\frac{7}{3}$	$-\frac{5}{3}$	$\frac{5}{3}$

Restricted tableau 2

	x_5	x_2	x_4	b
x_3	-2	$-\frac{1}{3}$	1	0
x_1	3	2^+	-1	1
x_6	-1	1	0	1
	-2	1	-1	1

Restricted tableau 3

	x_5	x_1	x_4	b
x_3	$-\frac{3}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$
x_2	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
x_6	$-\frac{5}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	$-\frac{7}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

The last row except the last element (*viz.*, 1/2) is nonpositive and one artificial variable, *viz.*, x_6 is still in the basis with nonzero value 1/2. Hence the equations have no nonnegative solution. However, the solution is $x = [-1 \quad 1 \quad 1]^t$.

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Nomenclature

Symbol	*Meaning*
'$\leftarrow$'	is replaced by
$B = (b_{ij})$	$m \times n$ matrix
$y = (y_j)$	n-vector
$g = (g_i)$	m-vector
I_m	unit matrix of order m
$A = (a_{ij})$	$m \times (n + m)$ matrix, $A = (B, I_m)$
t	transpose

$x = (x_j)$	$(n + m)$-vector, $x = [y_1 \cdots y_n x_{n+1} \cdots x_{n+m}]^t$
$z = (z_j)$	m-vector
$b = (b_i)$	m-vector, $b = g$
$c = (c_j)$	$(n + m)$-vector
d_j	j-th element of the d_j-row,
	$d_j = c_{n+1}a_{1j} + c_{n+2}a_{2j} + \cdots + c_{n+m}a_{mj} - c_j$
lp	linear programming

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