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On the distribution of the eigenvalues of a matrix differential operator

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Abstract

The paper deals with the nature of the spectrum associated with a type of second-order matrix differential operator with certain boundary conditions. It is found that under certain conditions satisfied by the co-efficients of the differential system, the spectrum is discrete. Some results are then obtained giving distributions of the eigenvalues on the real axis. The method employed depends, among others, upon some of the ideas and techniques of E. C. Titchmarsh[7].

Key words : Differential operator, eigenvalue problem, Hilbert space, Dirichlet (Neumann) problem Spectrum–discrete, continuous, point continuous, Green's matrix, meromorphic function, pseudo-monotonic, variation of the eigenvalues, distribution of the eigenvalues, convex downwards.

1. Introduction

Let $I: a < x < b$ be an interval on the real line: $a = -\infty$, $b = \infty$ or both being allowed. Let $C^0(I) = C(I)$ be the set of all real-valued continuous functions on I and $C^k(I)$, $k = 1, 2, \ldots$, denote the set of those $f \in C(I)$ for which $f^{(k)} \in C(I)$.

Consider the differential operator

$$M = \begin{pmatrix} -D^2 + p & q \\ q & -D^2 + r \end{pmatrix}, D \equiv \frac{d}{dx}, \tag{1.1}$$

where $p, q, r \in C^1(I)$; p, q, r are absolutely continuous over any compact sub-interval of I for $x \in I$.

Let the basic Hilbert space be $\mathcal{H} = \mathcal{L}^2(a, b)$ and let $\mathcal{D}$ represent the set of all $f \equiv \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \{f_1, f_2\} \in \mathcal{H}$ such that (i) $f \in C^1(I)$; (ii) f' absolutely continuous on every compact sub-interval of I: $f' \in$ A.C and (iii) $Mf \in \mathcal{H}$. We say that $f \in \mathcal{D}_0$ if $f \in \mathcal{H}$ satisfies the conditions (i) and (ii).

Let $\mathcal{C}$ be the set of complex numbers and $\lambda \in \mathcal{C}$; then

$$M\phi = \lambda\phi, \tag{1.2}$$

where $\phi = \binom{u}{v} = \{u, v\}$, $\phi \in \mathfrak{D}$, with some prescribed boundary conditions, gives rise to an eigenvalue problem both in the finite as well as in the singular case considered by Chakravarty[1, 2].

Let $a < \alpha < x < \beta < b$ and let the solution $\phi = \{u, v\}$ of (1 2) satisfy at α and β either

$$u(\alpha) = v(\alpha) = 0$$
$$u(\beta) = v(\beta) = 0 \tag{1.3}$$

or

$$u'(\alpha) = v'(\alpha) = 0$$
$$u'(\beta) = v'(\beta) = 0. \tag{1.4}$$

The eigenvalue problems (1.2)–(1.3) and (1.2)–(1.4) will henceforth be designated as the *Dirichlet problem* and the *Neumann problem* respectively over the interval (α, β). We can, without loss of generality, choose $\alpha = 0$.

The purpose of the present paper is to obtain certain conditions on p, q, r so that the spectrum of the given differential system may be discrete over I and then to obtain certain estimates giving the distribution of the eigenvalues on I.

The spectrum $\sigma(\lambda)$ of the system (1.2)–(1.3) or (1.2)–(1.4) may be defined as the set of λ values contributing to the expansion formula. It has been established by Chakravarty[2] (p. 403), that there exists at least a pair of linearly independent $\mathcal{L}^2$ solutions of the system (1.2)–(1.3) or (1.2)–(1.4) given by

$$\psi_k(x, \lambda) = m_{k1}(\lambda)\,\phi_1 + m_{k2}(\lambda)\,\phi_2 + \theta_k, \quad m_{kj} = m_{jk}, \tag{1.5}$$

where $\phi_j \equiv \phi_j(0/x, \lambda)$, $j = 1, 2$ are the "boundary condition vectors" at $x = 0$ (for definition see Chakravarty[1], p. 137) and $\theta_j \equiv \theta_j(0/x, \lambda)$ are determined from $[\phi_i, \theta_j] = \delta_{ij}$, $[\theta_1, \theta_2] = 0$, δ_{ij}, the Kronecker delta; ϕ_j, θ_j entire functions of λ.

By closely following the analysis given in Chaudhuri and Everitt[3] (pp. 95–119), it can be shown that the spectrum of the given system may be characterised by the properties of the matrix

$$(m_{ij}) \equiv \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{1.6}$$

If $\lambda = \mu + i\nu$, then the spectrum is

(i) *discrete*, if and only if $m_{ij}(\lambda)$ are all meromorphic, *i.e.*, the matrix (m_{ij}) is meromorphic:

(ii) *continuous*, if and only if $\lim_{\nu \to 0} \text{im}\, m_{ij}(\lambda)$ tends to a continuous, non-vanishing function, bounded for all $\mu \in (\mu_1, \mu_2)$; and

(iii) *point-continuous*, if and only if $\lim_{\nu \to 0} m_{ij}(\lambda)$ tends to infinity, but $\lim_{\nu \to 0} \text{im}\, m_{ij}(\mu)$ is a continuous, non-vanishing function in $N'(\mu)$, the deleted neighbourhood of μ. Finally, μ does not belong to the spectrum, if and only if $\lim_{\nu \to 0} \text{im}\, m_{ij}(\lambda) = 0$.

In discussions involving the eigenvalue problems, Green's matrix plays a very prominent role. The discussion of the Green's matrix for the finite integral (a, b) occurs in Chakravarty[1] (p. 148). For the singular case the Green's matrix is defined by

$$G(x, y, \lambda) = \begin{cases} \mathcal{G}(x, y, \lambda), & y < x \\ \mathcal{G}^T(y, x, \lambda), & y > x \end{cases}$$

where $\mathcal{G}(x, y, \lambda)$ is the matrix with elements $G_{ij}(x, y, \lambda) = (\psi_j^T(x, \lambda), A_l(y))$, the inner product of the vector $\psi_j^T(x, \lambda)$ (the transpose of $\psi_j(x, \lambda)$) and the lth column vector $A_l(y)$ of

$$A(y) = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}, \{x_j, y_j\} = \phi_j = \phi_j(0/y, \lambda),$$

the boundary condition vector at $y = 0$. See Chakravarty[2] (p. 403). Also Sen Gupta[6] (p. 91).

It follows that since ϕ_j, θ_j are entire functions of λ, $G(x, y, \lambda)$ is meromorphic, if and only if, the matrix (1.6) is meromorphic. This property of the Green's matrix will be utilised in our discussion. The present analysis depends upon the ideas and techniques as developed by Titchmarsh[7] for problems of second-order partial differential equations and employed by Chaudhuri and Everitt[4] (pp. 185–209) in solving corresponding problems on a type of fourth-order differential equations.

2. Notations

In what follows we use the following notations.

The accent denotes differentiation with respect to x;

P stands for the matrix $P = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$;

P_j being that in which the elements p, q, r are replaced by p_j, q_j, r_j;

$(f, g) = f_1 g_1 + f_2 g_2$, the inner product of the two vectors $f = \{f_1, f_2\}$, $g = \{g_1, g_2\}$;

$\langle y, z\rangle_{a,b} = \int_a^b (y, z)\, dt$, $\|y\|_{a,b} = \langle y, y\rangle_{a,b}$;

$(F, G, P) = f_1' g_1' + f_2' g_2' + p f_1 g_1 + r f_2 g_2 + q f_1 g_2 + q f_2 g_1$,

where

$F = \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} = \begin{pmatrix} f \\ f' \end{pmatrix}$, $f' = \{f_1', f_2'\}$, with a similar notation for G.

$D_{\mathcal{E}}(f, g) = D_{\mathcal{E}}(f, g, P) = \int_a^b (F, G, P)\, dt$, $\mathcal{E} = (a, b)$;

When $a = 0$, we write $D_b(f, g) = D_b(f, g, P)$ for $D_{\mathcal{E}}(f, g, P)$; $D_b(f) = D_b(f, f, P)$.

When $\mathcal{E} = [0, \infty)$, we define $D(f, g) = D(f, g, P) = \lim_{b\to\infty} D_b(f, g, P)$ and $D(f) = D(f, f, P)$.

If $\mathcal{E} = (x_s, x_t)$, where x_s, x_t are points on the real axis, we write $D_{s,t}(f, g) = D_{s,t}(f, g, P)$ for $D_{\mathcal{E}}(f, g, P)$.

E represents the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We note that if $p > 0$ and $\det P \geqslant 0$, $D_b(f)$ is always positive, (F, F, P) being positive definite.

3. Properties of $D_{\mathcal{E}}(f, g)$ for the finite interval $[0, b]$

Let $\lambda_n = \lambda_n(b)$ and $\psi_n(x) = \psi_n(b, x)$ denote respectively the eigenvalue and the eigenvector for the Dirichlet (Neumann) problem enunciated in Art. 1. Then some of the properties of $D_{\mathcal{E}}(f, g)$ are contained in the following lemmas.

Lemma 3.1: For the Dirichlet (Neumann) problem,

(i) $D_b(\psi_m, \psi_n) = \lambda_n \delta_{mn}$, where δ_{mn} is the Kronecker delta.

(ii) $D_b(\alpha\psi_m + \beta\psi_n) = \begin{cases} \alpha^2 \lambda_m + \beta^2 \lambda_n, & m \neq n \\ (\alpha + \beta)^2 \lambda_n, & m = n \end{cases}$; α, β constants.

Lemma 3.2: If $p > c$ and $\det(P - cE) \geqslant 0$, c, a positive real constant, then the eigenvalues for the Dirichlet (Neumann) problem are greater than or equal to c.

We have

$$D_b(\psi_n) = D_b(\psi_n, \psi_n, P) = D_b(\psi_n, \psi_n, P - cE) + c\int_0^b |\psi_n|^2\, dt.$$

The lemma follows, since the integral in the first expression on the right is positive definite and

$$\int_0^b |\psi_n|^2 \, dt = 1.$$

Lemma 3.3: Let (i) $f(x) = \{f_1, f_2\} \in \mathfrak{D}_0$ (ii) $f(0) = f(b) = 0$. Then if

$$c_m = \int_0^b (\psi_m, f) \, dt$$

be the Fourier co-efficient of f for the Dirichlet problem,

$$D_b(f, \psi_m) = \lambda_m c_m. \tag{3.1}$$

If further (iii) $p > 0$ and det $P \geqslant 0$, then

$$D_b(f) \geqslant \sum_{n=0}^{\infty} \lambda_n c_n^2. \tag{3.2}$$

Results (3.1) and (3.2) also hold for the Neumann problem, but now the condition (ii) is not required.

Since, on integration by parts

$$D_b(f, \psi_m) = \int_0^b (M\psi_m, f) \, dt + [(\psi_m', f)]_0^b,$$

(3.1) follows on utilizing $M\psi_m = \lambda_m \psi_m$.

To prove (3.2) we observe that by virtue of the condition (iii), $D_b(F) \geqslant 0$ for every vector $F \in \mathfrak{D}$. Hence

$$0 \leqslant D_b[f - \sum_{n=0}^{m} c_n \psi_n] = D_b[f - \sum_{n=0}^{m} c_n \psi_n, \; f - \sum_{n=0}^{m} c_n \psi_n, \; P]$$

$$= D_b(f) - 2 \sum_{n=0}^{m} c_n D_b(f, \psi_n) + \sum_{n=0}^{m} c_n^2 D_b(\psi_n) + 2 \sum_{r \neq m} c_n c_m D_b(\psi_m, \psi_n)$$

$$= D_b(f) - \sum_{n=0}^{m} \lambda_n c_n^2, \text{ by lemma 3.1 (i) and the result (3.1).}$$

(3.2) therefore follows.

Lemma 3.4: If (i) $f \in \mathfrak{D}_0$ (ii) $f(0) = f(b) = 0$ and if c_n is the Fourier co-efficient of f, then

$$D_b(f) = \sum_{n=0}^{\infty} \lambda_n c_n^2.$$

Let $\tilde{c}_n$ be the Fourier co-efficient of $\tilde{f} = Mf$.

Then by the Parseval theorem,

$$\int_0^b (f, \tilde{f})\, dt = \sum_{n=0}^{\infty} c_n \tilde{c}_n = \sum_{n=0}^{\infty} \lambda_n c_n^2,$$

where $\tilde{c}_n = \lambda_n c_n$, by Chakravarty[1], Lemma 3 (p. 150).

By integration by parts,

$\int_0^b (f, \tilde{f})\, dt = -(f, f')]_0^b + D_b(f, f, P) = D_b(f)$, by the condition (ii). The result therefore follows.

If d_n be the Fourier co-efficient of $g(x) \equiv \{g_1, g_2\}$, where g satisfies conditions similar to those of f in the above lemma, it follows similarly that

$$D_b(f, g) = \sum_{n=0}^{\infty} \lambda_n c_n d_n.$$

4. Extension to infinite interval (Spectrum assumed wholly discrete)

The Dirichlet and the Neumann problem for the infinite interval $[0, \infty)$ takes respectively the form

$$\left.\begin{array}{l} M\phi = \lambda\phi \\ u(0) = v(0) = 0 \end{array}\right\} \tag{4.1}$$

and

$$\left.\begin{array}{l} M\phi = \lambda\phi \\ u'(0) = v'(0) = 0 \end{array}\right\} \tag{4.2}$$

where

$$I: \; 0 \leqslant x < \infty.$$

The eigenvector $\psi_n \equiv \{\psi_{1n}, \psi_{2n}\}$ corresponding to the eigenvalue λ_n is integrable square at infinity.

It follows by integration by parts and using the relation

$$M\psi_n = \lambda_n \psi_n$$

where necessary, that

$$\int_0^{\xi} (\psi_n, \psi_n'')\, dt = [(\psi_n, \psi_n')]_0^{\xi} - \int_0^{\xi} |\psi_n'|^2\, dt.$$

Substituting for ψ_n'' from the differential system it follows, for both the Dirichlet and the Neumann problem, that

$$D_\xi(\psi_n, \psi_n, P) = (\psi_n(\xi), \psi_n'(\xi)) + \lambda_n \int_0^\xi |\psi_n|^2 dt.$$

Integrating first with respect to ξ over $(0, X)$ and then again integrating the result so obtained with respect to X over $(0, R)$, we obtain, after some easy reductions, that

$$\int_0^R (\hat{\psi}_n, \hat{\psi}_n, P)\left(1 - \frac{t}{R}\right)^2 dt = \frac{1}{R^2}\int_0^R |\psi_n|^2 dt + \lambda_n \int_0^R \left(1 - \frac{t}{R}\right)^2 |\psi_n|^2 dt,$$

where

$$\hat{\psi}_n = \begin{pmatrix} \psi_n \\ \psi_n' \end{pmatrix}.$$

By making R tend to infinity, it follows that $D(\psi_n) = \lambda_n$.

Similarly

$$D(\psi_m, \psi_n) = 0, \text{ if } m \neq n.$$

Let $\mathfrak{D}_1$ be associated with the Hilbert space $\mathcal{H}_1 = \mathcal{L}^2[0, \infty)$ in the same way as $\mathfrak{D}$ is associated with $\mathcal{H} = \mathcal{L}^2(a, b)$.

Then

$$\int_0^R \left(1 - \frac{t}{R}\right)(f, M\psi_n)\, dt = \int_0^R \left(1 - \frac{t}{R}\right)(F, \hat{\psi}_n, P)\, dt - \frac{1}{R}\int_0^R (f, \psi_n')\, dt,$$

$f \in \mathfrak{D}_1$, [$f(0) = 0$ for the Dirichlet problem].

It follows, on making R tend to infinity, that

$$D(f, \psi_n) = \lambda_n c_n,$$

where c_n is the Fourier co-efficient of f.

If, moreover, $p \geqslant 0$, det $P \geqslant 0$,

$$D(f) \geqslant \sum_{n=0}^{\infty} \lambda_n c_n.$$

We say that $p, q, r \in \mathcal{M}$, if p, q, r satisfy the conditions similar to those stated in Chakravarty[2] (Theorem II, p. 404), *viz.*,

(i) either $|p+r|+|q| \leqslant Q(x)$ or $|p|, |q|, |r| \leqslant Q(x)$

where $Q(x) \in C^1(I)$, $I: 0 \leqslant x < \infty$ and $Q(x) \geqslant \delta > 0$;

(ii) $\lim\limits_{x\to\infty} |Q'(x)/Q^c(x)| < \infty$, $c \leqslant 3/2$;

(iii) $F(x) = \int^x \{Q(t)\}^{-1/2}\, dt$ tends to infinity as x tends to infinity.

Or,

if p, q, r satisfy (i) where $Q(x)$ is continuous, monotone non-decreasing and $\int^{\infty} \{Q(2t)\}^{-1/2}\, dt$ divergent.

If $p, q, r \in \mathcal{M}$, we have $\tilde{c}_n = \lambda_n c_n$, where $\tilde{c}_n$ is the Fourier co-efficient of $\tilde{f} = Mf$ (Chakravarty[2], p. 413).

It follows by the Parseval theorem that

$$\int_0^\infty (f, \tilde{f})\, dx = \sum_{n=0}^{\infty} \lambda_n c_n^2.$$

Now, as before,

$$\int_0^R \left(1 - \frac{t}{R}\right)(f, \tilde{f})\, dt = \int_0^R \left(1 - \frac{t}{R}\right)(F, F, P)\, dt - \frac{1}{R} \int_0^R (f, f')\, dt,$$

$$f(0) = 0.$$

Therefore by making R tend to infinity, it follows that

$$D(f) = \int_0^\infty (f, \tilde{f})\, dt. \tag{4.3}$$

Hence

$$D(f) = \sum_{n=0}^{\infty} \lambda_n c_n^2, \text{ if } f \in \mathfrak{D}_1,\ p,\ q,\ r \in \mathcal{M} \text{ and } f(0) = 0.$$

If, in addition, $g \in \mathfrak{D}_0$, $g(0) = 0$, it follows in a similar manner that

$$D(f, g) = \sum_{n=0}^{\infty} \lambda_n c_n d_n,$$

where d_n is the Fourier co-efficient of g.

5. Extension to the case when the spectrum is possibly continuous

We define the H-matrix by

$$H(x, y\,\mu) = \begin{cases} \lim\limits_{\nu \to 0} \int\limits_0^{\mu} \operatorname{im} G(x, y, \lambda)\, d\sigma, & \mu > 0; \\ -\lim\limits_{\nu \to 0} \int\limits_0^{\mu} \operatorname{im} G(x, y, \lambda)\, d\sigma, & \mu < 0; \\ 0, & \mu = 0; \end{cases}$$

where $\lambda = \sigma + i\nu$ and $G(x, y, \lambda)$ is the Green's matrix in the singular case $[0, \infty)$. Then closely following the analysis as given in Titchmarsh[7] (pp. 41–55), it follows that:

Each element of $H(x, y, \mu) \in \mathcal{L}^2 [0, \infty)$, for fixed x.

$$F(x, \mu, f) = F(x, \mu) = \{F_1(x, \mu),\ F_2(x, \mu)\} = \int\limits_0^{\infty} H^T(y, x, \mu)\, f(y)\, dy \in \mathcal{L}^2 (0, \infty)$$

for every μ, if $f \in \mathcal{L}^2 [0, \infty)$.

If

$$J_b(f, g, \mu) = \frac{1}{\pi} \langle F(y, \mu, f),\ g(y) \rangle_{0,\, b}, \quad J(f, g, \mu) = \frac{1}{\pi} \langle F(y, \mu, f),\ g(y) \rangle_{0,\, \infty}$$

where

$$g \in \mathcal{L}^2 [0, \infty) \text{ and } J(f, \mu) = J(f, f, \mu),$$

then $J(f, \lambda)$ is non-decreasing;

$$| J(f, g, \beta) - J(f, g, \alpha) |^2 \leqslant \| f \|_{0,\, \infty} \| g \|_{0,\, \infty}. \tag{5.1}$$

Also, if $f \in \mathfrak{D}_1$, p, q, $r \in \mathcal{M}$, $g \in \mathfrak{D}_0$, then

$$\langle \tilde{f}, g \rangle_{,\, \infty} = \int\limits_{-\infty}^{\infty} \lambda\, dJ(f, g, \lambda), \quad \lambda \text{ real}. \tag{5.2}$$

[For discussion of H-matrix in detail, see Tiwari[8]].

Let $f \in \mathfrak{D}_0$ and choose b so that $0 < x < b < X$ and

$$f_X = \begin{cases} \left(1 - \dfrac{x}{X}\right) f, & x < X; \\ 0\ , & \text{otherwise.} \end{cases}$$

Then

$$f_X \in \mathfrak{D}_0$$

and

$$\int\limits_{\lambda_0}^{\infty} \lambda\, dJ_b(f_X, \lambda) \leqslant D_b(f_X), \ (\lambda, \text{ real}), \tag{5.3}$$

where λ_0 (independent of b) is the lower bound of the spectrum and conditions of lemma 3.3 are satisfied. (Compare Titchmarsh[7], pp. 95–96.)

Since

$$|J(f,\lambda)-J(f_X,\lambda)| \leqslant |J(f,f-f_X,\lambda)| + |J(f_X,f-f_X,\lambda)|$$
$$\leqslant \|f\|^{\frac{1}{2}}_{0,\infty} \|f-f_X\|^{\frac{1}{2}}_{0,\infty} + \|f_X\|^{\frac{1}{2}}_{0,\infty} \|f-f_X\|^{\frac{1}{2}}_{0,\infty},$$

[by (5.1) with $\alpha=0$, $\beta=\lambda$], it follows that $J(f_X,\lambda)$ tends to $J(f,\lambda)$ uniformly with respect to λ as X tends to infinity.

By computing $D_b(f_X)$ in a straight-forward manner, making X tend to infinity first and then b tend to infinity, we obtain

$$D_b(f_X) \text{ tends to } D(f), \text{ as } X, b \text{ tend to infinity.}$$

Therefore from (5.3), for any positive $\Lambda > \lambda_0$, we obtain

$$\int_{\lambda_0}^{\Lambda} \lambda dJ(f,\lambda) \leqslant D(f)$$

and since Λ is arbitrary,

$$\int_{\lambda_0}^{\infty} \lambda d\, J(f,\lambda) \leqslant D(f) \tag{5.4}$$

holds, where $f \in \mathfrak{D}_1$.

Let

$$f(0)=0,\ f\in\mathfrak{D}_1,\ \ p,q,r\in\mathcal{M},$$

then by (4.3) and (5.2) with $g=f$, we have

$$D(f) = \int_{-\infty}^{\infty} \lambda dJ(f,\lambda). \tag{5.5}$$

6. Variation of the eigenvalues with p, q, r.

Definition: The matrix $P = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$ is said to be *Pseudo-monotonic* in I, if $p > 0$, $\det P \geqslant 0$ in I and for $j > k$, $j, k = 0, 1, 2, \ldots$, $p_j \geqslant p_k$, $\det(P_j - P_k) = \det(P_k - P_j) \geqslant 0$, where p_s, q_s, r_s are the values of p, q, r at a point $x_s \in I$ and P_s is the matrix P with p, q, r replaced by p_s, q_s, r_s.

The matrix P may be called the *matrix* of the Dirichlet (Neumann) problem under consideration.

Let λ_n, ψ_n, c_n and $N(\lambda, P_j)$ denote, respectively, the eigenvalue, the eigenvector, the Fourier co-efficient and the number of eigenvalues not exceeding λ for the Dirichlet (Neumann) problem with matrix P_j and μ_n, χ_n, d_n and $N(\lambda, P_k)$ those for the same problem with matrix P_k.

We establish the following theorem.

Theorem 6.1. Let the matrix P be Pseudo-monotonic. Then

$$\lambda_n \leqslant \mu_n \quad \text{and} \quad N(\lambda, P_k) \geqslant N(\lambda, P_j), \quad j > k. \quad j, k = 0, 1, 2 \ldots .$$

Case I. Interval finite: Since P is Pseudo-monotonic, therefore for $j > k$, j, $k = 0, 1, 2, \ldots$, $p_j \geqslant p_k > 0$, det P_j, det $P_k \geqslant 0$ and det $(P_j - P_k) \geqslant 0$. Then by lemma 3.2 each eigenvalue λ_n is positive.

Now

$$D_b(f, P_k) = \int_0^b (F, F, P_k)\, dt, \quad F = \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} = \begin{pmatrix} f \\ f' \end{pmatrix}$$

$$= \int_0^b (F, F, P_j)\ dt + \int_0^b (F, F, P_k - P_j)\ dt.$$

Since $(F, F, P_k - P_j)$ is positive definite, therefore $D_b(f, P_k) \geqslant D_b(f, P_j)$ (6.1)
for any $f \in \mathfrak{D}_0$.

Put

$$f = \chi_0(x) = \{\chi_{10}, \chi_{20}\}.$$

Then

$$\| \chi_0 \|_{0, b} = 1 \text{ and we have}$$

$$\lambda_0 = \lambda_0 \| \chi_0 \|_{0, b} = \lambda_0 \sum_{n=0}^{\infty} c_n^2 \leqslant \sum_{n=0}^{\infty} \lambda_n c_n^2 \leqslant D_b(f, P_j) \leqslant D_b(f, P_k),$$

by (3.2) and (6.1).

Thus

$$\lambda_0 \leqslant \mu_0.$$

Put

$$f = \delta_0 \chi_0(x) + \delta_1 \chi_1(x),$$

where

δ_0, δ_1 are constants:

$$\delta_0 = B(A^2 + B^2)^{-1/2}, \quad \delta_1 = -A(A^2 + B^2)^{-1/2},$$

where

$$A = (\chi_0, \psi_0)_{0,b}, \; B = (\chi_1, \psi_0)_{0,b}, \; \delta_0^2 + \delta_1^2 = 1, \; c_0 = A\delta_0 + B\delta_1 = 0.$$

Then

$$\sum_{n=1}^{\infty} c_n^2 = \| \delta_0 \chi_0 + \delta_1 \chi_1 \|_{0,b} = \delta_0^2 + \delta_1^2 = 1.$$

Therefore

$$\lambda_1 = \lambda_1 \sum_{n=1}^{\infty} c_n^2 \leqslant \sum_{n=0}^{\infty} \lambda_n c_n^2 \leqslant D_b(f, P_j) \leqslant D_b(f, P_k), \quad f = \delta_0 \chi_0 + \delta_1 \chi_1.$$

Thus by lemma 3.1,

$$\lambda_1 \leqslant \delta_0^2 \mu_0 + \delta_1^2 \mu_1 \leqslant (\delta_0^2 + \delta_1^2) \mu_1,$$

showing that

$$\lambda_1 \leqslant \mu_1.$$

The general case $\lambda_n \leqslant \mu_n$ follows in the same way as Titchmarsh[7] (pp. 89–90). The second part of the problem is an immediate consequence of the first.

Case II. Interval infinite: When the interval $[0, b]$ is replaced by $[0, \infty)$, the theorem follows by exactly similar arguments as before by using the results of Art. 4.

Case III. Each spectrum possibly continuous but each has at its left hand end point a discrete eigenvalue λ_0 and μ_0 respectively.

We have, if χ_0 is the eigenvector corresponding to μ_0,

$$\lambda_0 = \lambda_0 \| \chi_0 \|_{0,\infty} = \lambda_0 \int_{\lambda_0}^{\infty} dJ(\chi_0, P_j, \lambda) \leqslant \int_{\lambda_0}^{\infty} \lambda dJ(\chi_0, P_j, \lambda)$$

$$\leqslant D(\chi_0, P_j) \leqslant D(\chi_0, P_k) = \mu_0.$$

Hence as before the result can be extended to other discrete eigenvalues. The theorem is therefore completely established.

7. Variation of the eigenvalues with the interval: upper and lower bounds of the nth eigenvalue

In the following we assume $p > 0$ and $\det P \geqslant 0$.

Let $N_X(\lambda)$ denote the number of eigenvalues not exceeding λ of the Dirichlet (Neumann) problem of the interval $[0, X]$. The following theorem holds.

Theorem 7.1. Let λ_n, μ_n denote, respectively, the nth eigenvalue for the Dirichlet (Neumann) problem of the interval $[0, b]$ and $[0, B]$, where $B > b$. Then

$$\lambda_n \geqslant \mu_n \quad \text{and} \quad N_b(\lambda) \leqslant N_B(\lambda).$$

Let ψ_n, c_n be the nth eigenvector and the Fourier co-efficient for the problem of the interval $[0, b]$ and χ_n, d_n those for the problem of the interval $[0, B]$.

Put

$$f(x) = \begin{cases} \psi_0(x), & 0 \leqslant x < b\,; \\ 0 \quad , & b \leqslant x \leqslant B. \end{cases}$$

Then by (3.2), it follows that $D_B(f) \geqslant \sum_{n=0}^{\infty} \mu_n d_n^2$ and therefore

$$\lambda_0 = D_b(\psi_0) = D_B(f) \geqslant \sum_{n=0}^{\infty} \mu_n d_n^2 \geqslant \mu_0 \sum_{n=0}^{\infty} d_n^2 = \mu_0,$$

showing that the result holds when $n = 0$. The case $\lambda_n \geqslant \mu_n$ for all integral values of n follows as before. The second part of the theorem is an obvious consequence of the first.

To obtain the bounds of the nth eigenvalue of the problem under discussion, we subdivide the fundamental interval $[0, X]$ into a finite number of mutually disjoint sub-intervals I_s : $[x_{s-1}, x_s]$, $s = 1, 2, \ldots, m$, $x_0 = 0$, $x_m = X$, and consider the Dirichlet and the Neumann problem for each sub-interval I_s.

For our problem of the interval $[0, X]$, let λ_n, ψ_n, c_n denote, respectively, the eigenvalue, the eigenvector and the Fourier co-efficient, the corresponding entities for the Neumann problem of the interval I_s being $\mu_{n,s}$, $\chi_{n,s}$, $d_{n,s}$ respectively. For the Dirichlet problem of the interval I_s, let $\lambda_{n,s}$ be the eigenvalue and $\psi_{n,s}$ the corresponding eigenvector.

Put

$$\mu'_n = \{\mu_{n,s};\ s = 1, 2, \ldots, m;\ n = 0, 1, 2, \ldots\},\ \mu'_0 \leqslant \mu'_1 \leqslant \mu'_2 \leqslant \ldots,$$

$$\lambda'_n = \{\lambda_{n,s};\ s = 1, 2, \ldots, m;\ n = 0, 1, 2, \ldots\},\ \lambda'_0 \leqslant \lambda'_1 \leqslant \lambda'_2 \leqslant \ldots,$$

Finally, suppose that

$M_s(\lambda)$ denote the number of eigenvalues not exceeding λ of the Neumann problem of the interval I_s;

$M'_X(\lambda)$, the number of numbers μ'_n not exceeding λ in the fundamental interval $[0, X]$.

$N_s(\lambda)$, the number of eigenvalues not exceeding λ of the Dirichlet problem of the interval I_s;

and $N'_X(\lambda)$, the number of numbers λ'_n not exceeding λ in $[0, X]$.

The following theorem is now established.

Theorem 7.2. With notations explained as above,

(i) $\mu_n' \leqslant \lambda_n \leqslant \lambda_n'$;

(ii) $N_X'(\lambda) \equiv \sum_{s=1}^{m} N_s(\lambda) \leqslant N_X(\lambda) \leqslant M_X'(\lambda) \equiv \sum_{s=1}^{m} M_s(\lambda)$.

We prove (i). Then (ii) is an immediate consequence of (i).

Put

$$f(x) = \psi_0(x),\ 0 \leqslant x \leqslant X.$$

Then by (3.2),

$$D_{s-1,s}(\psi_0) \geqslant \sum_{n=0}^{\infty} \mu_{n,s}\, d_{n,s}^2.$$

Hence

$$\lambda_0 = D_X(\psi_0) = \sum_{s=1}^{m} D_{s-1,s}(\psi_0) \geqslant \sum_{s=1}^{m} \sum_{n=0}^{\infty} \mu_{n,s}\, d_{n,s}^2$$

$$\geqslant \mu_0' \sum_{s=1}^{m} \sum_{n=0}^{\infty} d_{n,s}^2 = \mu_0' \sum_{s=1}^{m} \| \psi_0 \|_{x_{s-1}, x_s}, \text{ (by the Parseval theorem)}$$

Thus

$$\lambda_0 \geqslant \mu_0' \| \psi_0 \|_{0, X} = \mu_0'.$$

Put

$$f(x) = a_0 \psi_0(x) + a_1 \psi_1(x),\quad a_0^2 + a_1^2 = 1.$$

Then

$$\lambda_1 = \lambda_1 (a_0^2 + a_1^2) \geqslant \lambda_0 a_0^2 + \lambda_1 a_1^2 = D_X(a_0 \psi_0 + a_1 \psi_1)$$

$$= \sum_{s=1}^{m} D_{s-1,s}(a_0 \psi_0 + a_1 \psi_1) \geqslant \sum_{s=1}^{m} \sum_{n=1}^{\infty} \mu_{n,s}\, d_{n,s}^2,$$

where we have chosen $\mu_{0,s} = 0$, by suitably choosing the constants a_0, a_1 as in theorem 6.1.

Thus

$$\lambda_1 \geqslant \mu_1' \| a_0 \psi_0 + a_1 \psi_1 \|_{0, X} = \mu_1'.$$

The general case $\lambda_n \geqslant \mu_n'$ now follows as in theorem 6.1. The first part of the inequality is thus proved.

To prove the second part of the inequality (i), put

$$f(x) = \psi_{0,k}, \quad \text{in } x_{k-1} < x < x_k\,;$$
$$= 0, \quad \text{otherwise};$$

and

$$\lambda'_0 = \lambda_{0,k} \text{ for fixed } k,\ k = 1, 2, \ldots, m.$$

Then

$$\lambda'_0 = \lambda_{0,k} D_{k-1,k}(\psi_{0,k}) = D_X(f) \geqslant \sum_{n=0}^{\infty} \lambda_n c_n^2, \text{ by } (3.2)$$
$$= \lambda_0 \| \psi_{0,k} \|_{0,X} = \lambda_0.$$

Put

$$\lambda'_1 = \lambda_{j,k},\ j = 0,\ 1;\ k,\ \text{fixed}.$$

If $j = 1$, we put

$$\left.\begin{aligned} f(x) &= a_0\psi_{0,k}(x) + a_1\psi_{1,k}(x), \quad x_{k-1} < x < x_k \\ &= 0, \quad \text{otherwise} \end{aligned}\right\} a_0^2 + a_1^2 = 1.$$

If $j = 0$, we take

$$\left.\begin{aligned} f(x) &= a\psi_{0,k}\ , \quad x_{k-1} < x < x_k \\ &= b\psi_{0,m}\ , \quad x_{m-1} < x < x_m \\ &= 0 \quad\ , \quad \text{otherwise} \end{aligned}\right\} a^2 + b^2 = 1.$$

The analysis now proceeds as in theorem 6.1 and Chaudhuri and Everitt[1], (pp. 196–197), so as to obtain $\lambda'_1 \geqslant \lambda_1$ and for any positive integral n, $\lambda'_n \geqslant \lambda_n$. The second part of the inequality (i) is thus proved. Hence the theorem follows.

Let (p_1, q_1, r_1), (p_2, q_2, r_2) be the values of (p, q, r) at the points $x = x_{s-1}$ and $x = x_s$, respectively, of the sub-interval I_s: $[x_{s-1}, x_s]$. Also, let P reduce to P_1, at $x = x_{s-1}$ and to P_2 at $x = x_s$.

Let

$$N_{iX}(\lambda) = N_X(\lambda,\ p_i,\ P_i),\ \ i = 1,\ 2;$$

$N_2(\lambda, s)$, the number of eigenvalues not exceeding λ of the Dirichlet problem of the interval I_s with matrix P_2;

and

$M_1(\lambda, s)$, the number of eigenvalues not exceeding λ of the Neumann problem of the interval I_s with matrix P_1.

We establish the following theorem.

Theorem 7.3. Let the matrix P be Pseudo-monotonic. Then with notations explained above

$$\sum_{s=1}^{n} N_2(\lambda, s) \leqslant N_X(\lambda) \leqslant \sum_{s=1}^{n} M_1(\lambda, s), \quad \text{for fixed } n,$$

where $X > Y$, Y being a root of det $(P - \lambda E) = 0$, for fixed λ.

Since P is Pseudo-monotonic, therefore for $0 < p_1 \leqslant p \leqslant p_2$,

$$\det (P - P_2) \geqslant 0, \ \det (P_2 - P_1) \geqslant 0.$$

Now, for two positive quadratic forms $\Sigma\, c_{ik}\, x_i\, x_k$, $\Sigma\, d_{ik}\, x_i\, x_k$, the inequality

$$| c_{ik} |^{1/n} + | d_{ik} |^{1/n} \leqslant | c_{ik} + d_{ik} |^{1/n} \tag{7.1}$$

holds, where $| c_{ik} |$ are determinants of the co-efficients, n positive integer (Hardy Littlewood, Polýa[5], p. 35, Formula 2.13.8).

Since

$$P_2 - \lambda E = (P - \lambda E) + (P_2 - P), \quad P_1 - \lambda E = P_1 - P_2 + P_2 - \lambda E,$$

it follows from (7.1), since P is Pseudo-monotonic, that

$$\det (P_2 - \lambda E), \ \det (P_1 - \lambda E) \geqslant 0, \text{ if } \det (P - \lambda E) \geqslant 0. \tag{7.2}$$

By theorem 6.1 it follows that

$$N_{2X}(\lambda) \leqslant N_X(\lambda) \leqslant N_{1X}(\lambda). \tag{7.3}$$

Also by theorem 7.2,

$$\sum_{s=1}^{m} N_2(\lambda, s) \leqslant N_{2X}(\lambda), \quad \sum_{s=1}^{m} M_1(\lambda, s) \geqslant N_{1X}(\lambda) \tag{7.4}$$

From (7.3) and (7.4),

$$\sum_{s=1}^{m} N_2(\lambda, s) \leqslant N_X(\lambda) \leqslant \sum_{s=1}^{m} M_1(\lambda, s). \tag{7.5}$$

Let us choose λ so that $p_1 > \lambda$, $\det (P_1 - \lambda E) \geqslant 0$, which, by (7.2) holds if $p > \lambda$, $\det (P - \lambda E) \geqslant 0$. Then by lemma 3.2, there are no eigenvalues less than λ with this choice of λ and therefore

$$M_1(\lambda, s) = 0 \quad \text{and} \quad N_2(\lambda, s) = 0, \tag{7.6}$$

whenever $p > \lambda$, $\det (P - \lambda E) \geqslant 0$.

Let Y be determined as the root of $\det \left(P(Y) - \lambda E\right) = 0$, where λ is a given real number. Since p is increasing, it is possible to choose $x > Y$ so that $p > \lambda$ holds. For all such x, since P is Pseudo-monotonic, $\det \left(P(x) - P(Y)\right) \geqslant 0$ and therefore $P(x) - \lambda E = P(Y) - \lambda E + P(x) - P(Y)$, by (7.1) leads to

$$\det \left(P(x) - \lambda E\right) \geqslant 0.$$

Let the interval $[0, X]$, $X > Y$, be chosen large enough so that for a point of subdivision x_n, say, for some $n < m$, $x_n = Y$ holds. Then (7.6) holds for all $s > n$, and the theorem follows.

Since λ is given, Y is fixed and therefore n is fixed. It follows therefore from the above theorem that $N_X(\lambda)$ is bounded independently of X. Since by theorem 7.1, $N_X(\lambda)$ increases with X, therefore

$$\lim_{X\to\infty} N_X(\lambda) = N(\lambda),$$

where, as will be evident from discussion in Art. 8 next, $N(\lambda)$ ($<\infty$), represents th number of eigenvalues not less than λ in the singular case.

8. A criterion for the discreteness of the spectrum

The following theorem provides a criterion for the discreteness of the spectrum of the boundary value problem under consideration.

Theorem 8.1. Let (i) p, q, r satisfy the conditions laid down in Art. 1, the matrix P being Pseudo-monotonic. If (ii) $p > \alpha \geqslant 0$, det $(P - \alpha E) \geqslant 0$, then the spectrum is discrete over the range (α, β).

Let λ_{nX}, $\lambda_{nX'}$ denote the eigenvalues for the problems of the intervals $[0, X]$ and $[0, X']$ respectively. Then by theorem 7.1, for $X \leqslant X'$, $\lambda_{nX} \geqslant \lambda_{nX'}$, showing that $\{\lambda_{nX}\}$ is steadily decreasing. Now by condition (ii) $\lambda_{nX} \geqslant \alpha$. Thus $\{\lambda_{nX}\}$ tends to a limit λ_n, say, as X tends to infinity. Hence the sequence $\{\lambda_{jX}\}$, $j = 0, 1, \ldots, h$, of eigenvalues lying in (α, β) tend to $\{\lambda_j\}$, $j = 0, \ldots, h$, (not necessarily all different), as X tends to infinity.

Let $\lambda_0 < \lambda_1$. Since the Green's matrix $G(X, x, \xi, \lambda)$, $\lambda = \mu + i\nu$, is regular except for simple poles at λ_{nX}, therefore $G(X, x, \xi, \lambda)$ is regular if $\lambda_0 + \delta \leqslant \mu \leqslant \lambda_1 - \delta$, where $\delta = 1/4\,(\lambda_1 - \lambda\,)$ and X large enough. (Compare Titchmarsh[7], p. 149).

We introduce the matrix $H(x, y)$ which is not a Green's matrix but has the same discontinuity property as the Green's matrix for the 'X-Case', substitute

$$\Gamma_{ij}(X, x, \xi, \lambda) = G_{ij}(X, x, \xi, \lambda) - H_{ij}(x, \xi),$$

G_{ij}, H_{ij} elements of G and H respectively,

and argue as in Chakravarty[2] (pp. 401–402), so as to obtain

$$|\Gamma_{ij}(X, x, \xi, \lambda)| \leqslant (\nu^{-2} + 1)^{1/2} \quad K(x, \xi, \delta, |\lambda|),$$

where K is a constant depending on the arguments shown.

Thus

$$|G_{ij}(X, x, \xi, \lambda)| \leqslant M\,|\nu|^{-1}$$

for given

$$x,\ \xi,\ x \neq \xi,\ \lambda_0 + \delta \leqslant \mu \leqslant \lambda_1 - \delta,\ \ -\delta \leqslant \nu \leqslant \delta.$$

Then by arguments similar to those in Titchmarsh[7] (p. 149), it follows that the Green's matrix $G(x, \xi, \lambda)$ in the singular case $[0, \infty)$ is regular except at the points λ_n and that λ_n is at most a simple pole of $G(x, \xi, \lambda)$. Hence the spectrum is discrete over (α, β).

Again, from above it follows that $G(x, \xi, \lambda)$ is a meromorphic function of λ and therefore the matrix $(m_{rs}(\lambda))$ is also meromorphic (*vide*, Art. 1). Hence also the spectrum is discrete over (α, β).

Finally, defining $f(x)$ by

$$f(x) = \psi_{0, X}, \quad 0 < x < X$$
$$= 0, \qquad \text{otherwise,}$$

and following Titchmarsh[7] (p. 150), by using (5.3), it can be shown that λ_0 is actually an eigenvalue. In the general case, λ_n is an eigenvalue for the boundary value problem in the singular case $[0, \infty)$.

$$N(\lambda) = \lim_{X \to \infty} N_X(\lambda)$$

is thus the number of eigenvalues not less than λ in the singular case $[0, \infty)$.

In particular, if p, q, r satisfy the conditions of Art. 1 and the matrix P is Pseudo-monotonic, the spectrum is discrete over $(0, \beta)$.

9. Distribution of the eigenvalues

Put

$$\triangle = \alpha + \gamma + \{(\gamma - \alpha)^2 + 4\beta^2\}^{1/2}$$

and

$$\eta = \alpha + \gamma - \{(\gamma - \alpha)^2 + 4\beta^2\}^{1/2}$$

where α, β, γ are real numbers and λ is a real number, $\lambda \geqslant 1/2 \max(\triangle, \eta)$.

We seek for solutions of the equation

$$\left.\begin{aligned} u''(x) - \beta v(x) + (\lambda - \alpha) u(x) = 0 \\ v''(x) - \beta u(x) + (\lambda - \gamma) v(x) = 0 \end{aligned}\right\} \tag{9.1}$$

where $\{u, v\}$ satisfy the Dirichlet-form of boundary conditions, *viz.*,

$$u(0) = 0 = v(0); \quad u(X) = 0 = v(X). \tag{9.2}$$

Solving (9.1) for u, v and making $\{u, v\}$ satisfy the boundary conditions (9.2), we derive after some easy steps

$$\sin \xi X \sin \xi X = 0 \tag{9.3}$$

where

$$\xi^2 = \lambda - \tfrac{1}{2}\eta, \quad \zeta^2 = \lambda - \tfrac{1}{2}\triangle.$$

Therefore, if $N_X(\lambda, \alpha, \beta, \gamma)$ be the number of eigenvalues not exceeding λ in the interval $(0, X)$, we have

$$N_X(\lambda, \alpha, \beta, \gamma) \geqslant \frac{X}{\pi}\left[\left(\lambda - \frac{1}{2}\triangle\right)^{1/2} + \left(\lambda - \frac{1}{2}\eta\right)^{1/2}\right] - 2. \tag{9.4}$$

Similarly, if $M_X(\lambda, \alpha, \beta, \gamma)$ be the number of eigenvalues not exceeding λ in the interval $(0, X)$ of (9.1) with boundary conditions in Neumann's form, *viz*.,

$$u'(0) = 0 = v'(0); \quad u'(X) = 0 = v'(X) \tag{9.5}$$

we have

$$M_X(\lambda, \alpha, \beta, \gamma) \leqslant \frac{X}{\pi}\left[\left(\lambda - \frac{1}{2}\triangle\right)^{1/2} + \left(\lambda - \frac{1}{2}\eta\right)^{1/2}\right] + 2 \tag{9.6}$$

Lemma 9.1. Let (i) $p > r$, (ii) p, q monotone increasing and

(iii) $(p - r)r' - 2qq' \geqslant 0$.

Then

$$\triangle(x) = p + r + \{(p - r)^2 + 4q^2\}^{1/2}$$

and

$$\eta(x) = p + r - \{(p - r)^2 + 4q^2\}^{1/2}$$

are both monotone increasing.

Since

$$(p - r)^2 + 4q^2 \geqslant 4q^2,$$

it follows that $\{(p - r)^2 + 4q^2\}^{1/2}$ is monotone increasing. Therefore

$$(p - r)(p' - r') + 4qq' \geqslant 0$$

and

$$\frac{d\eta(x)}{dx} \geqslant 0;$$

so that $\eta(x)$ is monotone increasing. Again, since $\{(p - r)^2 + 4q^2\}^{1/2} \geqslant p - r$, it follows that $\triangle(x) \geqslant 2p$. Therefore $\triangle(x)$ is monotone increasing.

The lemma remains true if $p - r$ is assumed monotone increasing instead of q.

We establish the following theorems on the distribution of the eigenvalues of the boundary value problem under consideration.

Theorem 9.1. Let the matrix P be Pseudo-monotonic and p, q, r satisfy the conditions of lemma 9.1. Then $N(\lambda)$, the number of eigenvalues not exceeding λ in the singular case of the problem under consideration, is given by

$$N(\lambda) = \frac{1}{\pi} \int_0^X \left[\left\{ \lambda - \frac{1}{2} \triangle(x) \right\}^{1/2} + \left\{ \lambda - \frac{1}{2} \eta(x) \right\}^{1/2} \right] dx$$

$$+ O(X^{1/2} \lambda^{1/4}), \quad \lambda \to \infty,$$

where X is determined by det $(P(X) - \lambda E) = 0$.

It follows from theorem 7.3, with notations explained there, that

$$\sum_{s=1}^{n} N_2(\lambda, s) \leqslant N_{X'}(\lambda) \leqslant \sum_{s=1}^{n} M_1(\lambda, s),$$

where $X' \geqslant X$ and X is given by det $(P(X) - \lambda E) = 0$.

Making X' tend to infinity through certain sequence, we then obtain

$$\sum_{s=1}^{n} N_2(\lambda, s) \leqslant N(\lambda) \leqslant \sum_{s=1}^{n} M_1(\lambda, s). \tag{9.7}$$

For the interval I_s: (x_{s-1}, x_s), let $\triangle_{js}(x)$, $\eta_{js}(x)$, $j = 1, 2$, stand for $\triangle(x)$ and $\eta(x)$ respectively when the matrix P is replaced by P_1 at $x = x_{s-1}$ and by P_2 at $x = x_s$. Then it follows from (9.4), (9.6) and (9.7) that

$$\sum_{s=1}^{n} \left[\left\{ \lambda - \frac{1}{2} \triangle_{2s}(x) \right\}^{1/2} + \left\{ \lambda - \frac{1}{2} \eta_{2s}(x) \right\}^{1/2} \right] \frac{\delta_s}{\pi} - 2n \leqslant N(\lambda)$$

$$\leqslant \sum_{s=1}^{n} \left[\left\{ \lambda - \frac{1}{2} \triangle_{1s}(x) \right\}^{1/2} + \left\{ \lambda - \frac{1}{2} \triangle_{1s}(x) \right\}^{1/2} \right] \frac{\delta_s}{\pi} - 2n \tag{9.7'}$$

where δ_s is the length of the interval I_s.

Noting that

$$F(x) = \{\lambda - \tfrac{1}{2} \triangle(x)\}^{1/2} + \{\lambda - \tfrac{1}{2} \eta(x)\}^{1/2}$$

by lemma 9.1, steadily decreases from $F_0 \equiv F(0)$ to $F_X \equiv F(X)$ as x increases from 0 to X, it is possible to choose the points of sub-division x_s of the interval $(0, X)$ in such a manner that the oscillation of $F(x)$ in each I_s is equal to

$$\frac{F_0 - F_X}{n}.$$

(Compare Chaudhuri and Everitt[4], p. 206 and De Wet and Mandl[9], pp. 572–580.)

Thus in I_s,

$$\left[\left\{\lambda - \frac{1}{2}\Delta_{1s}(x)\right\}^{1/2} + \left\{\lambda - \frac{1}{2}\eta_{1s}(x)\right\}^{1/2}\right] - \left[\left\{\lambda - \frac{1}{2}\Delta_{2s}(x)\right\}^{1/2} + \left\{\lambda - \frac{1}{2}\eta_{2s}(x)\right\}^{1/2}\right] = \frac{F_0 - F_X}{n}.$$

This leads to

$$\sum_{s=1}^{n}\left[\left\{\lambda - \frac{1}{2}\Delta_{1s}(x)\right\}^{1/2} + \left\{\lambda - \frac{1}{2}\eta_{1s}(x)\right\}^{1/2}\right]\frac{\delta_s}{\pi} \leqslant I(\lambda) + \frac{X(F_0 - F_X)}{n\pi} \tag{9.8}$$

where

$$I(\lambda) = \frac{1}{\pi}\int_X^0 \left[\left\{\lambda - \frac{1}{2}\Delta(x)\right\}^{1/2} + \left\{\lambda - \frac{1}{2}\eta(x)\right\}^{1/2}\right] dx.$$

Similarly,

$$\sum_{s=1}^{n}\left[\left\{\lambda - \frac{1}{2}\Delta_{2s}(x)\right\}^{1/2} + \left\{\lambda - \frac{1}{2}\eta_{2s}(x)\right\}^{1/2}\right]\frac{\delta_s}{\pi} \geqslant I(\lambda) - \frac{X(F_0 - F_X)}{n\pi} \tag{9.9}$$

Hence from (9.7′), (9.8) and (9.9),

$$|N(\lambda) - I(\lambda)| \leqslant \frac{X(F_0 - F_X)}{n\pi} + 2n. \tag{9.10}$$

Choose n so that the right hand side of (9.10) is minimum. This gives

$$n^2 = \frac{X(F_0 - F_X)}{2\pi}.$$

Therefore from (9.10),

$$N(\lambda) = I(\lambda) + O\{X^{1/2}(F_0 - F_X)^{1/2}\}. \tag{9.11}$$

The theorem follows from (9.11), since $|F_0 - F_e| \leqslant |F(0)| \leqslant K\lambda^{\frac{1}{4}}$, K, const.

The following theorem is next established.

Theorem 9.2. If the conditions of theorem 9.1 are satisfied and if (i) either p or (ii) r or (iii) $p+r$ or (iv) $p+q+r$, $(q(0) \geqslant 0)$, be convex downwards, then

$$N(\lambda) \sim \frac{1}{\pi} \int_0^X \left[\left\{ \lambda - \frac{1}{2} \triangle(x) \right\}^{1/2} + \left\{ \lambda - \frac{1}{2} \eta(x) \right\}^{1/2} \right] dx, \text{ as } \lambda \text{ tends to infinity.}$$

We give details of the proof when $(p+r)$ is convex downwards with outlines in other cases.

Since $p(x) + r(x)$ is convex downwards, we have

$$p(u) + r(u) \leqslant p(0) + r(0) + \frac{p(X) + r(X) - p(0) - r(0)}{X} u,$$

$$0 < u < X, \quad p(0), > 0, \quad r(0) \geqslant 0,$$

since $p > 0$, det $P \geqslant 0$ for x in I.

This leads to

$$\lambda - \frac{1}{2} \eta(u) \geqslant \lambda - \frac{1}{2} \{p(u) + r(u)\} \geqslant \lambda - \frac{1}{2} \{p(X) + r(X)\} \frac{u}{X},$$

so that

$$I(\lambda) = \frac{1}{\pi} \int_X^{\theta} \left[\left\{ \lambda - \frac{1}{2} \triangle(x) \right\}^{1/2} + \left\{ \lambda - \frac{1}{2} \eta(x) \right\}^{1/2} \right] dx$$

$$\geqslant \frac{1}{\pi} \int_0^X \left[\lambda - \frac{1}{2} \{p(X) + r(X)\} \frac{u}{X} \right]^{1/2} dx$$

$$\geqslant \frac{1}{\pi} X \lambda^{1/2} \left(1 - \frac{Q(X)}{2\lambda} \right)^{1/2}, \text{ where } Q(X) = p(X) + r(X). \tag{9.12}$$

Therefore from theorem 9.1 and the inequality (9.12), it follows that

$$|N(\lambda) - I(\lambda)| \leqslant K\pi X^{-1/2} \lambda^{-1/4} \left(1 - \frac{Q(X)}{2\lambda} \right)^{-1/2} I(\lambda) = \epsilon I(\lambda), \text{ say, where}$$

ϵ tends to zero as λ tends to infinity, X being determined by

$$\det (P(X) - \lambda E) = 0.$$

Thus the theorem is proved when $(p + r)$ is convex downwards.

Again, since

$$(p - r)^2 + 4q^2 \leqslant (p + r)^2 + 4q^2 \leqslant (p + r + 2q)^2,$$

therefore

$$\lambda - \frac{1}{2} \triangle (u) \geqslant \lambda - (p + q + r) \geqslant \lambda - \{p(X) + q(X) + r(X)\} \frac{u}{X} \;,$$

since $p + q + r$, $(q(0) \geqslant 0)$, is convex downwards: $0 < u < X$.

Finally,

since

$$\{(p - r)^2 + 4q^2\}^{1/2} \geqslant p - r, \;\; r - p,$$

it follows that

$$\lambda - \tfrac{1}{2} \eta (u) \geqslant \lambda - p(u), \;\; \lambda - r(u).$$

Therefore

$$\lambda - \frac{1}{2} \eta (u) \geqslant \lambda - p(X) \frac{u}{X}, \;\; 0 < u < X,$$

if p is convex downwards,

and

$$\lambda - \frac{1}{2} \eta (u) \geqslant \lambda - r(X) \frac{u}{X} \;, \; 0 < u < X,$$

if r is convex downwards.

In any case the analysis therefore follows as before.

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References

1. Chakravarty, N. K. Some problems in eigenfunction expansions (I). *Quart. J. Matl (Oxford)*, 1965, **16**, 135–150.

2. Chakravarty, N. K. Some problems in eigenfunction expansions (III). *Quart. . Math. (Oxford)*, 1968, **19**, 397–415.

3. CHAUDHURI, J. AND EVERITT, W. N. — On the spectrum of ordinary second order differential operators. *Proc. Roy. Soc. Edinb., A*, 1969, **68** (II) (7) 95–119.

4. CHAUDHURI, J. AND EVERITT, W. N. — The spectrum of a fourth order differential operator. *Proc. Roy. Soc. Edinb., A*, 1969, **68** (III) (13), 185–210.

5. HARDY, LITTLEWOOD AND PÓLYA — *Inequalities*, Cambridge, 1934, Ch. II, pp. 34–36.

6. SEN GUPTA PRABIR KUMAR — On the convergence of eigenfunction expansions associated with a vector-matrix differential equation. *Jour. Ind. Inst. Sc.*, 1978, **60** (A), 89–110.

7. TITCHMARSH E. C. — *Eigenfunction Expansions Associated with Second Order Differential Equations* Part II, Oxford University Press, 1958.

8. TIWARI, S. — *On Eigenfunction Expansions Associated with Differential Equations*, Calcutta University, Ph.D. Thesis, 1972.

9. DE WET, J. S. AND MANDL, F. — On the asymptotic distribution of eigenvalues. *Proc. Roy. Soc., A*, 1950, **200**, 572–580.