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Short Communication

Exact solution for the unsteady motion of a viscous fluid in a porous annulus

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Abstract

In this note, the problem of unsteady motion in a porous annulus has been studied. Using finite Hankel transform, closed form solution is obtained for the axial velocity component under the restriction that the ratio of suction and injection at the walls are same.

Key words: Unsteady, Viscous, Porous, Annulus, Suction, Injection.

Introduction

The geometry of porous layers is of great importance in the problems of pulmonary physiology, where there is a need for measuring the blood volume in the lungs.

The purpose of this note is to present the exact solution for the time dependent motion of a viscous fluid in an annulus with porous walls. It is assumed that the rate of suction at one wall is equal to the rate of injection at the other. Finite Hankel transform is applied and closed form solution for the axial velocity is obtained. The average axial velocity profiles are depicted graphically.

Basic equations

We consider the flow to take place in a porous annulus bounded by two infinite cylinders with radii a and b ($b > a$). A cylindrical polar coordinate system (r, θ, x) is chosen with axis of annulus as x axis, and v, u the radial and axial velocity components respectively. The governing equations are

$$\rho\left[\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}+u\frac{\partial v}{\partial x}\right]=-\frac{\partial p}{\partial r}$$

$$+\mu\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial x^2}-\frac{v}{r^2}\right] \tag{1}$$

$$\rho\left[\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial x}\right]=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}\right] \tag{2}$$

Continuity equation:

$$\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{\partial u}{\partial x}=0 \tag{3}$$

The boundary conditions are:

$$\left.\begin{array}{llll} \text{At } t=0, & u=0 & \text{for} & a\leq r\leq b \\ \text{At } r=a,\ b & u=0 & \text{for} & \text{all } t \\ \text{At } r=a & \multicolumn{3}{l}{v=V_a \text{ and } r=b,\ v=V_b} \end{array}\right\} \tag{4}$$

The condition that suction and injection rates are equal implies

$$aV_a=bV_b \tag{5}$$

This restriction makes the axial velocity independent of x.

Integrating (3) and (1), we get

$$v=\frac{aV_a}{r} \tag{6}$$

and

$$p=-\frac{\rho V_a^2 a^2}{2r^2}+A \tag{7}$$

where A is independent of r.

Introducing the following non-dimensional variables

$$\lambda=\frac{r}{a},\ \phi=\frac{u}{\dfrac{(p_0-p_L)a^2}{\mu L}},\ \tau=\frac{\mu t}{\rho a^2} \tag{8}$$

where L is the characteristic length, and (p_0-p_L) is the pressure difference, we obtain from (3) the equation for ϕ as

$$\frac{\partial\phi}{\partial\tau}=1+\frac{\partial^2\phi}{\partial\lambda^2}+\left(\frac{1-\alpha}{\lambda}\right)\frac{\partial\phi}{\partial\lambda} \tag{9}$$

where $\alpha = \rho V_0 \, a/\mu$ = Cross Reynold's number of suction parameter. The boundary conditions (4) become

$$\text{At } \tau = 0, \phi = 0 \text{ for } 1 \leq \lambda \leq b/a = \sigma$$

$$\text{At } \lambda = 1, \phi = 0 \text{ for all } \tau. \tag{10}$$

In obtaining this equation, we have tacitly assumed a non-zero axial pressure gradient which is necessary for setting up an unsteady flow. This in turn implies that A is a linear function of x. We can write the solution (9) as the sum of a steady part, and an unsteady part in the form

$$\phi(\lambda, \tau) = \phi_\infty(\lambda) - \phi_T(\lambda, \tau) \tag{11}$$

From (9), by taking the steady part only, we get the equation for ϕ_∞ as

$$1 + \frac{\partial^2 \phi_\infty}{\partial \lambda^2} + \left(\frac{1-\alpha}{\lambda}\right) \frac{\partial \phi_\infty}{\partial \lambda} = 0 \tag{12}$$

with boundary conditions

$$\phi_\infty = 0 \text{ at } \lambda = 1, \sigma.$$

The solution of (12) is

$$\varphi_\infty = A + B\lambda^\alpha - C\lambda^2 \text{ for } \alpha \neq 2$$

where

$$A = \frac{\sigma^2 - \sigma^\alpha}{2(2-\alpha)(1-\sigma^2)}, \quad B = \frac{1-\sigma^2}{2(2-\alpha)(1-\sigma^\alpha)}, \quad C = \frac{1}{4-2\alpha}$$

and

$$\phi_\infty = A_1(1-\lambda^2) - \frac{\lambda^2 \ln \lambda}{2} \text{ for } \alpha = 2 \tag{14}$$

where

$$A_1 = \frac{\sigma^2 \ln \sigma}{2(1-\sigma^2)}$$

The equation for ϕ_T is

$$\frac{\partial^2 \phi_T}{\partial \lambda^2} + \left(\frac{1-\alpha}{\lambda}\right) \frac{\partial \phi_T}{\partial \lambda} = \frac{\partial \phi_T}{\partial \tau} \tag{15}$$

with boundary conditions

At $\tau = 0$, $\phi = \phi_\infty$

At $\lambda = 1, \sigma, \phi_\tau = 0$

As $\tau \to \infty$, $\phi_\tau \to 0$.

By substituting $\phi_\tau = \lambda^{(a/2)} \psi$, (15) is simplified to

$$\frac{\partial^2\psi}{\partial\lambda^2} + \frac{1}{\lambda}\frac{\partial\psi}{\partial\lambda} - \left(\frac{a}{2\lambda}\right)^2 \psi = \frac{\partial\psi}{\partial\tau} \tag{16}$$

Now, we define (Sneddon[2]) the Hankel transform as

$$\bar{\psi}(s, \tau) = \int_1^\sigma \lambda\psi(\lambda, \tau)\, [J_k(s)\, Y_k(s\lambda) - J_k(s\lambda)\, Y_k(s)]\, d\lambda$$

where $k = a/2$, s is a root of the equation (Abromowitz and Stegun[3])

$$J_k(s)\, Y_k(s\sigma) - J_k(s\sigma)\, Y_k(s) = 0 \tag{17}$$

where J_k and Y_k are Bessel functions of first and second kind respectively. The inverse transform is

$$\psi(\lambda, \tau) = \frac{\pi^2}{2} \sum_s \frac{s^2 \left(J_k^2(s)\right) \bar{\psi}}{J_k^2(s) - J_k^2(s\sigma)} [J_k(s\lambda)\, Y_k(s) - J_k(s)\, Y_k(s\lambda)] \tag{18}$$

where summation is taken over the positive roots of (17). Applying the above transform in (16) and its inverse, we obtain the solution for axial velocity in dimensionless form as

$$\phi = \phi_\infty - \lambda^{(a/2)} \frac{\pi^2}{2} \sum_s \frac{(e^{-s^2\tau})\, \bar{\psi}\, s^2\, J_k^2(s\lambda)}{J_k^2(s) - J_k^2(s\sigma)} [J_k(s\lambda)\, Y_k(s) - J_k(s)\, Y_k(s\lambda)] \tag{19}$$

where ϕ_∞ is given by (13) and (14), and $\bar{\psi}$ is the transform of ψ. The series on the right side is convergent. It can be seen that as $\tau \to \infty$, we obtain solution for steady case as given by Berman[4]. It should be remarked that Verma and Gaur[5] have obtained a similar solution using Laplace transform, but it was commented that further analysis was not possible. However, in this note, the solution is more elegant and graphical representation is provided.

Figure 1 gives the average axial velocity profiles for various suction parameters. It can be seen that the suction parameter increases, the point of maximum velocity shifts towards the other boundary. Fig. 2 depicts the axial velocity and shows how the unsteady part dies out with passage of time. In Fig. 3, the average axial velocity is represented as a function of σ.

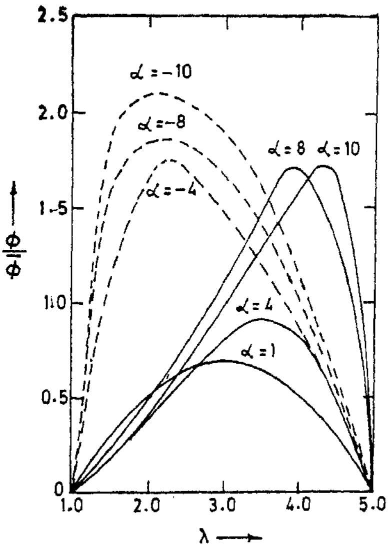

FIG. 1. Average axial velocity profiles for different suction parameters at $\tau = \pi$.

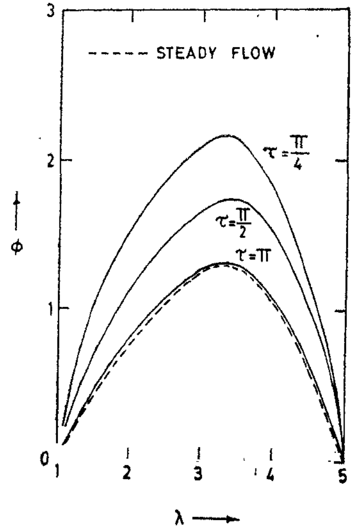

FIG. 2. Axial velocity field at different times when $\alpha = 4$, $0 = 5$.

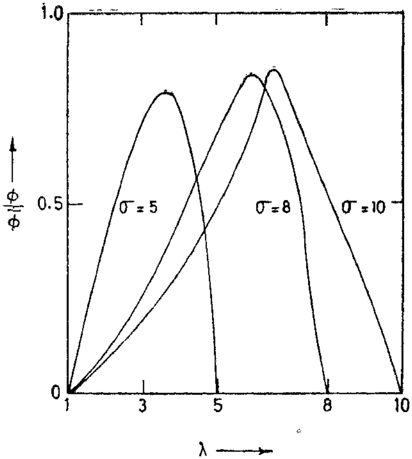

FIG. 3. Average axial velocity field for different values of σ when $\tau = \pi$ and $\alpha = 4$.

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