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Stress distribution in a semi-infinite strip with loadings on the long edges

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Abstract

The problem of a semi-infinite strip under normal loading on the long edges is solved in this paper and numerical results are given. Stress distributions in the strip under various lengths of loading on the long edges are presented. The results are compared with the limited ones available in the literature. Utility of the results in a practical problem in prestressed concrete is indicated.

Key words: Semi-infinite strip, stress analysis, normal loading on the long edges.

1. Introduction

Stress distribution in a long body under externally applied loads near the ends is a common problem in civil engineering, *e.g.*, the end block in a prestressed concrete beam. One such problem is treated in this paper. The problem is that of a long strip with loadings over a small length on the long edges while the narrow ends are traction free. In this case each half can be considered as a semi-infinite strip. A general solution by taking the stress function in terms of Fourier series and integrals, for the stress analysis of semi-infinite strips, has been given by Iyengar[1,2] wherein a number of references can be found on this problem. Later the same problem has been solved by Gupta[3] and Bogy[4].

The analytical solution presented here is based on the solution given by Iyengar[1] who has developed the stress function for a semi-infinite strip subjected to arbitrary loadings on all the three edges. Using this solution, numerical results have been obtained for various lengths of uniformly distributed load on the long edges and are given in this paper. The results have been compared with the available ones.

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2. Analysis

Consider a semi-infinite strip bounded in the right half-plane by the lines $x = 0$ and $y = \pm b$ as shown in Fig. 1. The strip is assumed to be in a state of generalized plane stress. The longitudinal edges of the strip are subjected to normal tractions symmetrical about the x-axis, while the transverse edge is kept free of tractions. The boundary conditions can then be written as

$$\sigma_x \big|_{x=0} = 0; \quad \tau_{xy} \big|_{x=0} = 0 \tag{1}$$

$$\sigma_y \big|_{y=\pm b} = -f(x); \quad \tau_{xy} \big|_{y=\pm b} = 0 \tag{2}$$

where

$$f(x) = \frac{P}{2c} \text{ for } 0 \leqslant x \leqslant 2c \tag{3}$$
$$= 0 \quad \text{for } x > 2c$$

P being the total load distributed over a length $2c$ on the long edge.

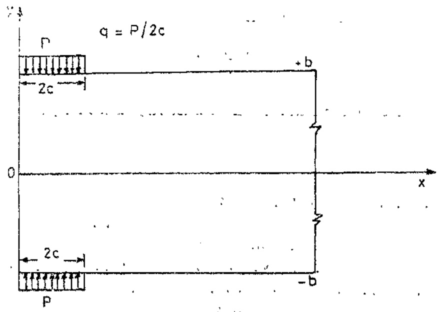

FIG. 1. Semi-infinite strip subjected to normal distributed loads at its end.

Considering the symmetry about x and y axes the following stress function is taken for this case from the stress function developed in ref. (1).

$$\Phi(x, y) = \sum_{m=1,2,3}^{\infty} A_m \frac{\cos(m\pi y/b)}{(m\pi/b)^2} \{1 + (m\pi x/b)\} e^{-(m\pi x/b)}$$

$$+ \int_0^{\infty} C(a) \frac{\cos ax}{a^2 \cosh ab} [ay \sinh ay - (1 + ab \coth ab) \cosh ay] \, da. \tag{4}$$

The above function satisfies biharmonic equation and hence the stress components are given by

$$\sigma_x = \frac{\delta^2 \Phi}{\delta y^2}, \sigma_y = \frac{\delta^2 \Phi}{\delta x^2}$$

and

$$\tau_{xy} = -\frac{\delta^2 \Phi}{\delta x \, \delta y}. \tag{5}$$

The coefficients A_m and function $C(a)$ are to be determined from the boundary conditions.

It can be seen that the shear stress boundary conditions are satisfied automatically

Satisfying the other boundary condition along transverse edge, namely

$$\sigma_x \big|_{x=0} = 0,$$

we get

$$-\sum_{m=1,2,3}^{\infty} A_m \cos\,(m\pi y/b) + \int_0^{\infty} \frac{C(a)}{\cosh ab}\, [ay \sinh ay$$

$$+ (1 - ab \coth ab) \cosh ay]\, da = 0. \tag{6}$$

By taking finite Fourier Transform and simplifying, eqn. (6) can be reduced to

$$A_m = 4\,(-1)^m \int_0^{\infty} \frac{(ab)\,(m\pi)^2 \,.\, \tanh ab}{[(ab)^2 + (m\pi)^2]^2}\, C(a)\, da. \tag{7}$$

On satisfying the remaining boundary condition, namely,

$$\sigma_y \big|_{y=\pm b} = -f(x), \text{ we get}$$

$$-f(x) = \sum_{m=1,2,3}^{\infty} A_m \cos m\pi \{(m\pi x/b) - 1\}\, e^{-(m\pi x/b)}$$

$$-\int_0^{\infty} C(a) \frac{\cos ax}{\cosh ab}\, [ab \sinh ab - (1 + ab \coth ab) \cosh ab]\, da.$$

Rewriting the above equation

$$\sum_{m=1,2,3}^{\infty} A_m (-1)^m \{1-(m\pi x/b)\} e^{-(m\pi x/b)} - f(x)$$

$$= \int_0^{\infty} C(a)\left[1+\frac{2ab}{\sinh 2ab}\right] \cos ax \, da. \tag{8}$$

Using the Fourier Integral Transformation, we get

$$C(a)\left[1+\frac{2ab}{\sinh 2ab}\right] = \sum_{m=1,2,3}^{\infty} \frac{2}{\pi} \int_0^{\infty} A_m (-1)^m \{1-(m\pi x/b)\}$$

$$e^{-(m\pi x/b)} \cos ax \, dx - \frac{2}{\pi} \int_0^{\infty} f(x) \cos ax \, dx. \tag{9}$$

Eqn. (9) simplifies to

$$C(a)\left[1+\frac{2ab}{\sinh 2ab}\right] = \frac{4b}{\pi} \sum_{m=1,2,3}^{\infty} A_m \frac{(-1)^m (m\pi)(ab)^2}{[(ab)^2+(m\pi)^2]^2} - F(a) \tag{10}$$

where

$$F(a) = \frac{2}{\pi} \int_0^{\infty} f(x) \cos ax \, dx \tag{11}$$

$$= \frac{P}{\pi ac} \sin 2ac. \tag{12}$$

$C(a)$ can be determined from eqn. (10) in terms of A_m and substitution of $C(a)$ in eqn. (7) gives the following infinite system to determine $A_m s$:

$$A_m = 16\pi^2 m^2 \sum_{r=1,2,3}^{\infty} (-1)^{r+m} r A_r K(r,m) - 4(-1)^m \pi^2 m^2 \beta_m \tag{13}$$

where

$$K(r,m) = \int_0^{\infty} \frac{(ab)^3 \cdot \tanh ab}{\left[1+\frac{2ab}{\sinh 2ab}\right][(ab)^2+(m\pi)^2]^2 [(ab)^2+(r\pi)^2]^2} \, bda \tag{14}$$

and

$$\beta_m = \int_0^\infty \frac{F(\alpha)\ \alpha b\ \tanh\ \alpha b}{\left[1 + \frac{2\alpha b}{\sinh\ 2\alpha b}\right][(\alpha b)^2 + (m\pi)^2]^2}\ d\alpha. \tag{15}$$

When substituted for $C(\alpha)$ the expressions for stresses are

$$\sigma_x = -\sum_{m=1,2,3}^{\infty} A_m \left[\cos(m\pi y/b)\{1 + (m\pi x/b)\}\, e^{-(m\pi x/b)} - m(-1)^m F_m\right] - a_0$$

$$\sigma_y = +\sum_{m=1,2,3}^{\infty} A_m \left[\cos(m\pi y/b)\{(m\pi x/b) - 1\}\, e^{-(m\pi x/b)} - m(-1)^m H_m\right] + c_0$$

$$\tau_{xy} = -\sum_{m=1,2,3}^{\infty} A_m \left[\sin(m\pi y/b)(m\pi x/b)\, e^{-(m\pi x/b)} - m(-1)^m U_m\right] - e_0 \tag{16}$$

where

$$F_m = 4b^3 \int_0^\infty \frac{[\alpha y \sinh\ \alpha y + (1 - \alpha b \coth\ \alpha b)\cosh\ \alpha y]}{[\cosh\ \alpha b + (\alpha b/\sinh\ \alpha b)]\,[(\alpha b)^2 + (m\pi)^2]^2}\, \alpha^2\ \cos \alpha x\, d\alpha$$

$$H_m = 4b^3 \int_0^\infty \frac{[\alpha y \sinh\ \alpha y - (1 + \alpha b \coth\ \alpha b)\cosh\ \alpha y]}{[\cosh\ \alpha b + (\alpha b/\sinh\ \alpha b)]\,[(\alpha b)^2 + (m\pi)^2]^2}\, \alpha^2 \cos \alpha x\, d\alpha$$

$$U_m = 4b^3 \int_0^\infty \frac{[\alpha y \cosh\ \alpha y - \alpha b \coth \alpha b \sinh \alpha y]}{[\cosh\ \alpha b + (\alpha b/\sinh\ \alpha b)]\,[(\alpha b)^2 + (m\pi)^2]^2}\, \alpha^2 \sin \alpha x\, d\alpha$$

$$a_0 = \int_0^\infty F(\alpha)\, \frac{[\alpha y \sinh \alpha y + (1 - \alpha b \coth \alpha b)\cosh \alpha y]}{[\cosh\ \alpha b + (\alpha b/\sinh \alpha b)]} \cos \alpha x\, d\alpha$$

$$c_0 = \int_0^\infty F(\alpha)\, \frac{[\alpha y \sinh\ \alpha y - (1 + \alpha b \coth\ \alpha b)\cosh \alpha y]}{[\cosh\ \alpha b + (\alpha b/\sinh\ \alpha b)]} \cos \alpha x\, d\alpha$$

$$e_0 = \int_0^\infty F(\alpha)\, \frac{[\alpha y \cosh\ \alpha y - \alpha b \coth \alpha b \sin h\, \alpha y]}{[\cosh \alpha b + (\alpha b/\sinh\ \alpha b)]} \sin \alpha x\, d\alpha. \tag{17}$$

3. Numerical solution

Stress components at any point in the semi-infinite strip can be determined using eqns. (16) where the values of A_ms are determined from eqn. (13). Eqn. (13) forms the necessary condition to be satisfied by A_ms such that all the boundary conditions are identically satisfied. However, eqn. (13) leads to an infinite set of simultaneous equations in A_ms. By considering only a finite number of terms in the series in the evaluation of stresses, the biharmonic equation is still exactly satisfied whereas the boundary conditions are approximately satisfied. However, it may be shown by determining the stresses with different number of terms that the boundary conditions have been satisfied to sufficient accuracy.

Theoretical treatment on the convergence aspects of such infinite series is given by Teoderescu[5] and Gupta[3] and in the book by Kantorovich and Krylov[6]. The details of calculations together with numerical method of finding the infinite integrals, etc., are given in ref. 1. It may be mentioned here that for the evaluation of the infinite integrals five point formula in a numerical integration scheme has been used.

4. Numerical results

Numerical results have been obtained for various values of $k = (c/b) = 1 \cdot 0,\ 0 \cdot 8,\ 0 \cdot 6,\ 0 \cdot 5,\ 0 \cdot 4,\ 0 \cdot 2$ and $0 \cdot 1$ using the IBM 360/44 computer. For each k, stress components are evaluated along various horizontal and vertical sections of the strip. While evaluat-

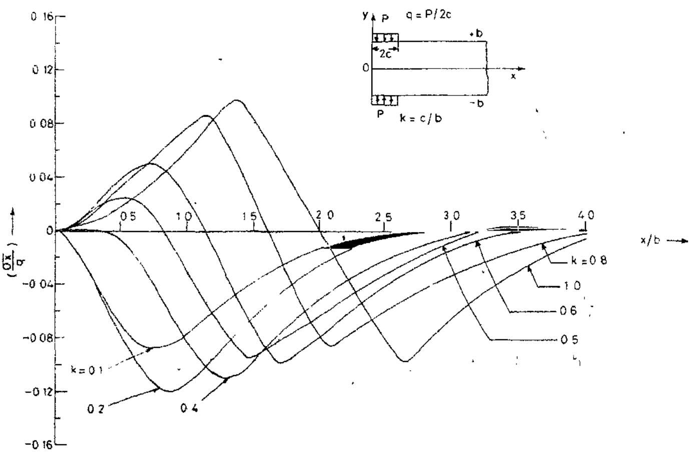

FIG. 2. Distribution of $\left(\frac{\sigma_x}{q}\right)$ for various values of k along $(y/b) = 0$.

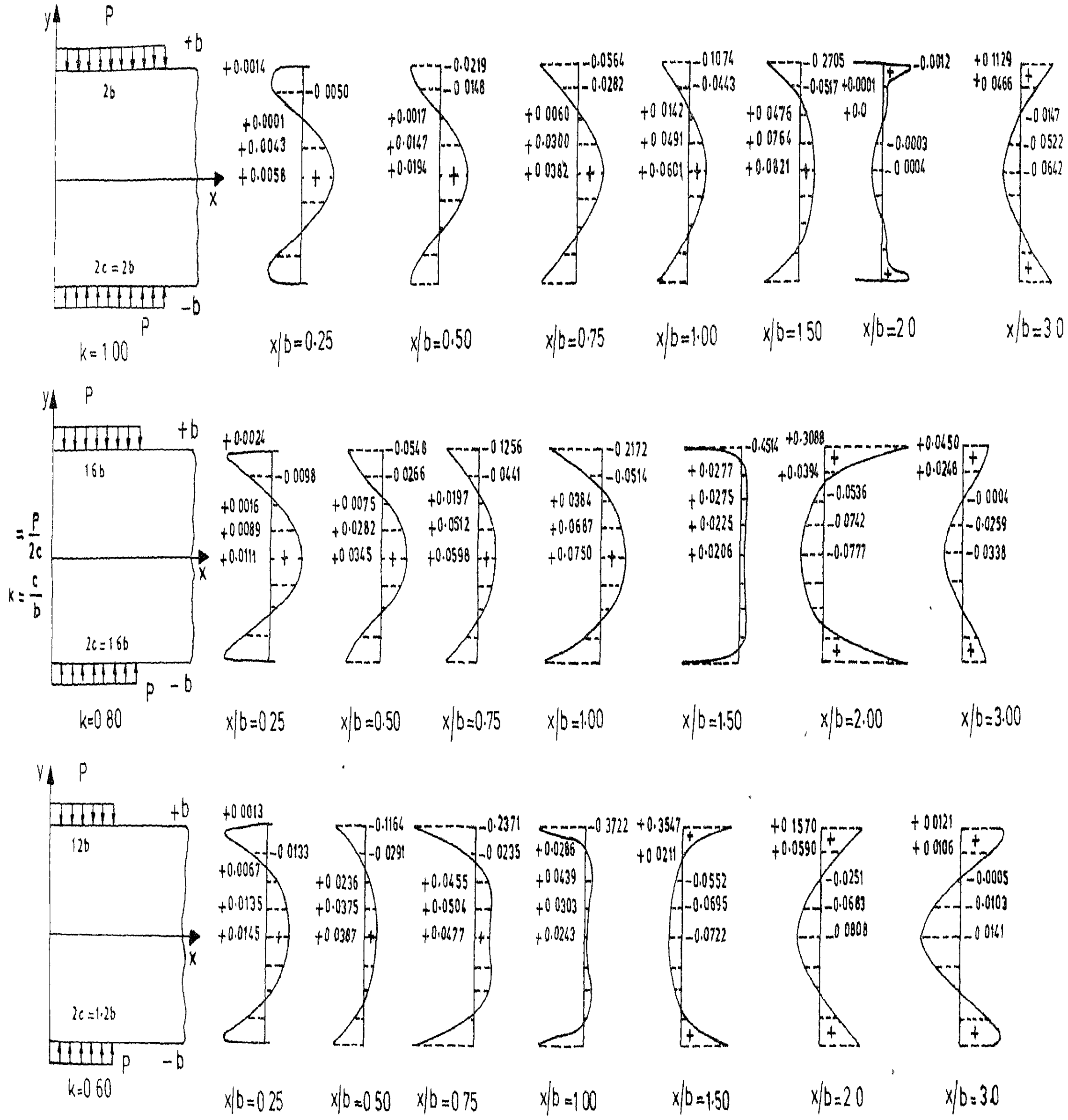

FIG 3. Distribution of $\left(\frac{\sigma^{\pi}}{q}\right)$ at various vertical sections for different values of k

ing the stresses, it has been found, that the number of terms to be considered in the series for a good convergence decreases with increasing value of x. Convergence is quite fast in the case of transverse and shear stresses when compared to that of the longitudinal stress.

The number of terms considered for various ranges of x are given below. For

(i) $x/b \geqslant 1$, 8 terms

(ii) $0{\cdot}3 \leqslant x/b < 1{\cdot}0$, 10 terms

(iii) $0{\cdot}2 \leqslant x/b < 0{\cdot}3$, 14 terms

and

(iv) $0 < x/b < 0{\cdot}2$, 18 terms

The exact shear stress boundary values are obtained automatically. Longitudinal boundary values of σ_y, obtained with finite number of terms in the series, are in good agreement with the actual values. Convergence of the series for σ_x along $x = 0$ is very slow and large number of terms should be considered to get results comparable to that of actual values.

Distribution of σ_x along $y = 0$ for varying k is shown in Fig. 2. Variation of σ_x over the depth for varying x/b values is shown in Fig. 3, for $k = 1{\cdot}0$, $0{\cdot}8$ and $0{\cdot}6$. The transverse stress distribution along $y/b = 0$, $0{\cdot}25$, $0{\cdot}50$ and $0{\cdot}75$ for 7 values of k is presented in Figs. 4, 5, 6 and 7 respectively. τ_{xy} distribution along $y/b = 0{\cdot}50$ is shown in Fig. 8.

Higuchi[7] has given σ_x and σ_y distributions for a k value of $2{\cdot}0$. He has evaluated the infinite integrals using the Cauchy theorem of residues in determining the stresses. Using the procedure indicated in the present paper the same distributions have been obtained and presented in Figs. 9 and 10. Both the results are found to be in very good agreement.

Bay[8], solving the same problem for concentrated load P using finite difference technique, has obtained σ_y distribution along $y = 0$ as shown in Fig. 11. This distribution is compared with that obtained for $k = 0{\cdot}001$ by the present procedure in Fig. 11. Since the total load P is distributed over a very small length, it could be approximated to a concentrated load. According to Bay, σ_y becomes zero at $x = 0{\cdot}8\,b$ whereas this value equals $0{\cdot}96\,b$ in the present method.

The transverse stress is compressive in nature initially over some distance greater than $2\,c$ and then changes to tensile in nature before reaching negligibly small value. The

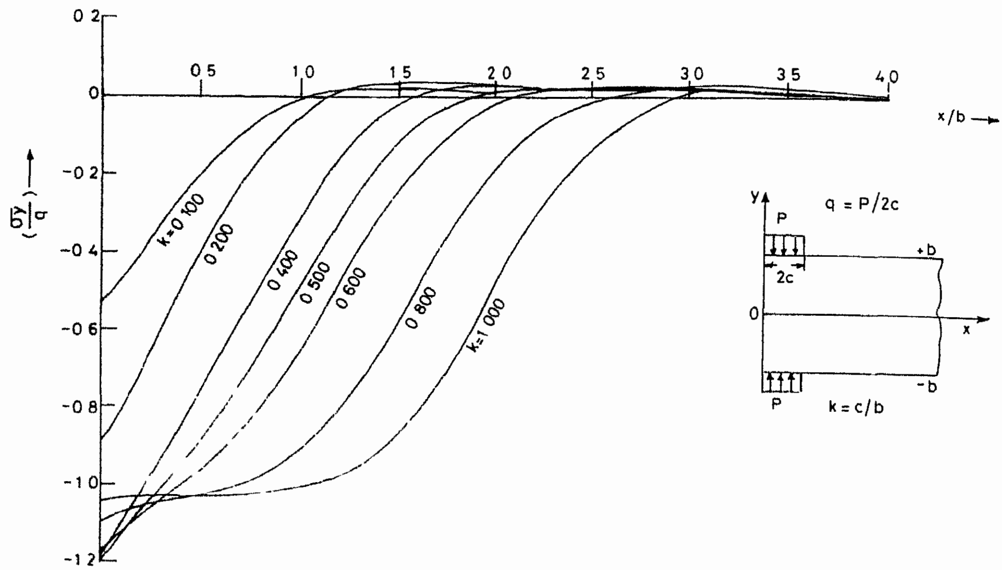

FIG 4 Distribution of $\left(\frac{\sigma_y}{q}\right)$ for different values of k along $(y/b) = 0$.

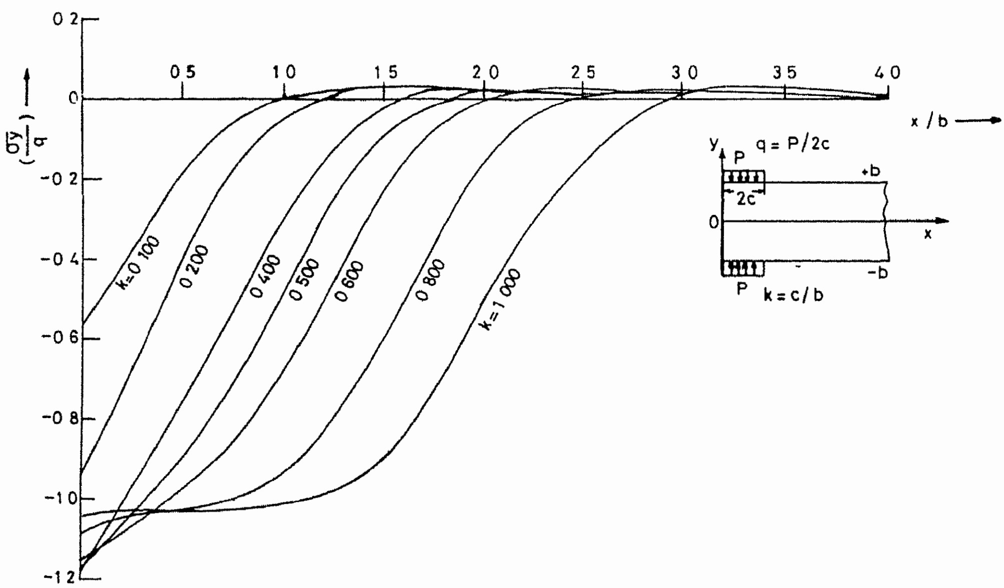

FIG. 5. Distribution of $\left(\frac{\sigma_y}{q}\right)$ for different values of k along $(y/b) = 0{\cdot}25$.

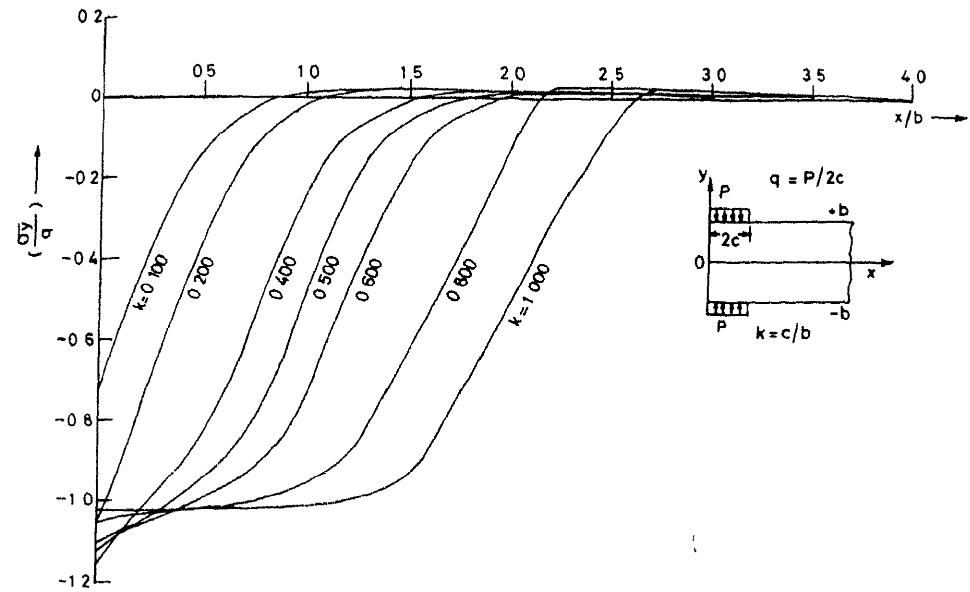

FIG. 6. Distribution of $\left(\frac{\sigma_y}{q}\right)$ for different values of k along $(y/b) = 0{\cdot}50$.

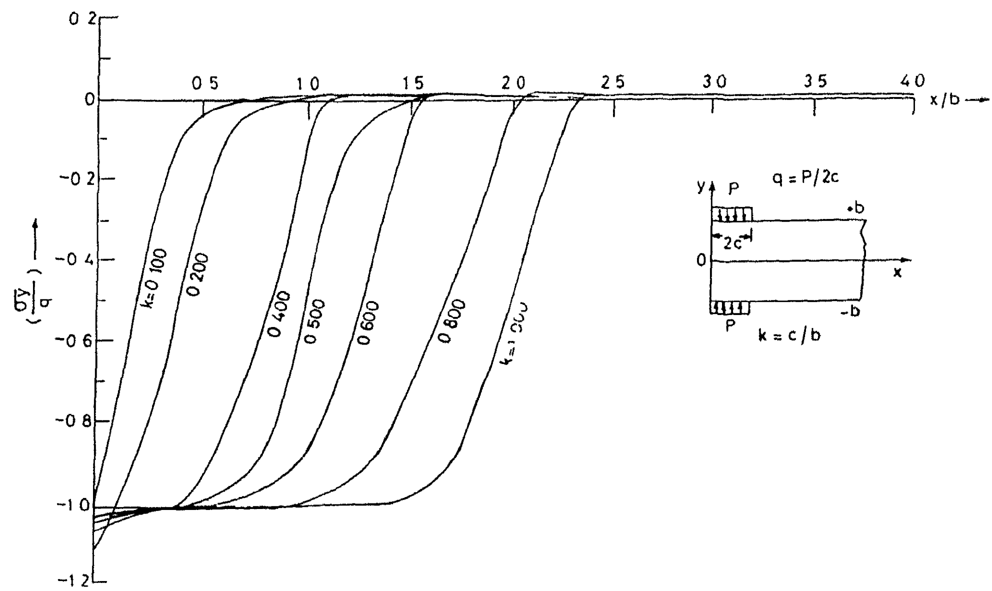

FIG 7. Distribution of $\left(\frac{\sigma_y}{q}\right)$ for different values of k along $(y/b) = 0{\cdot}75$.

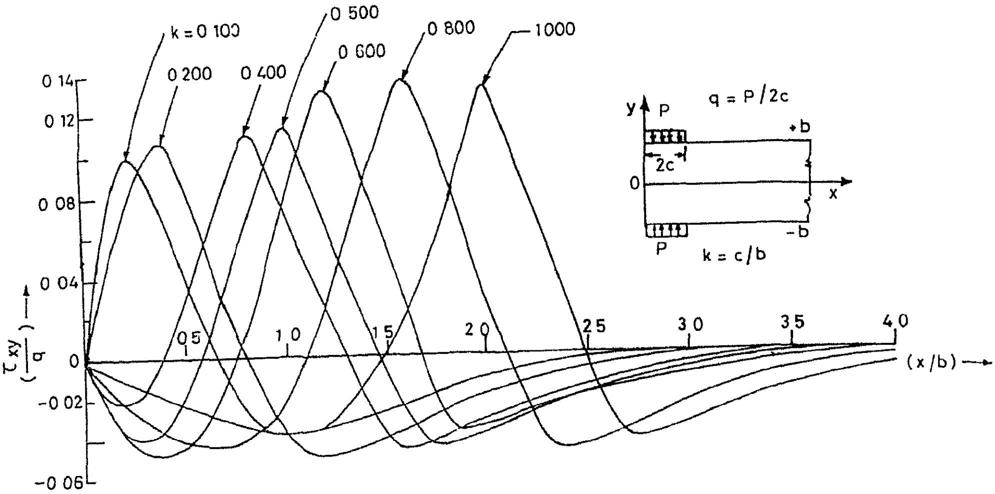

FIG 8 Distribution of $\left(\frac{\tau_{xy}}{q}\right)$ for different values of k along $(y/b) = 0\cdot50$.

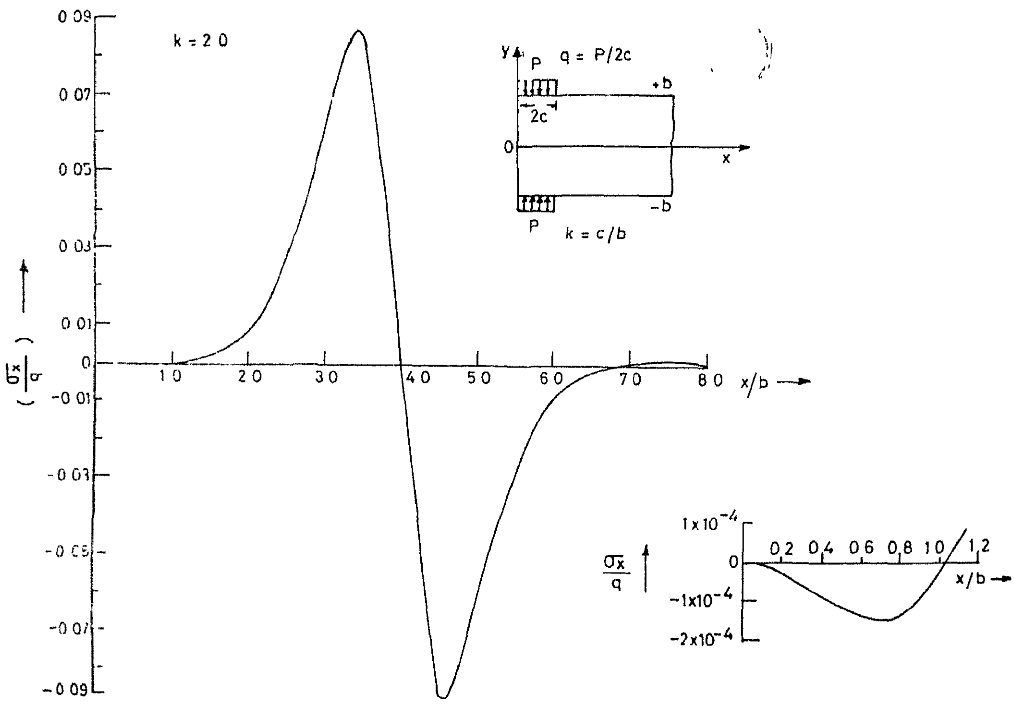

FIG 9. Distribution of $\left(\frac{\sigma_x}{q}\right)$ along $(y/b) = 0$.

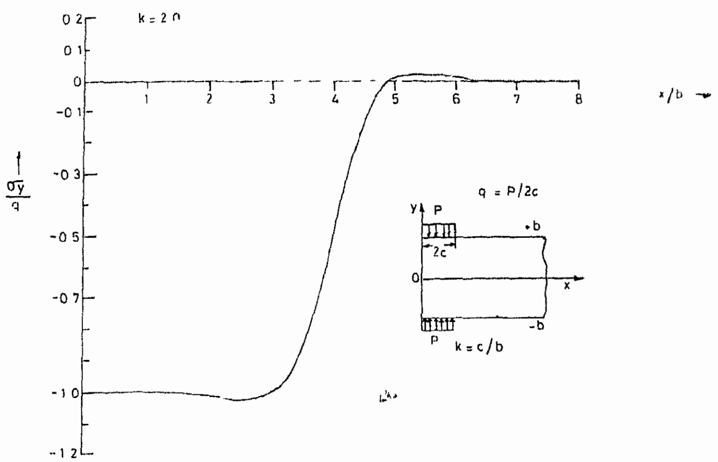

Fig 10 $\left(\frac{\sigma_y}{q}\right)$ distribution along $(y/b) = 0$

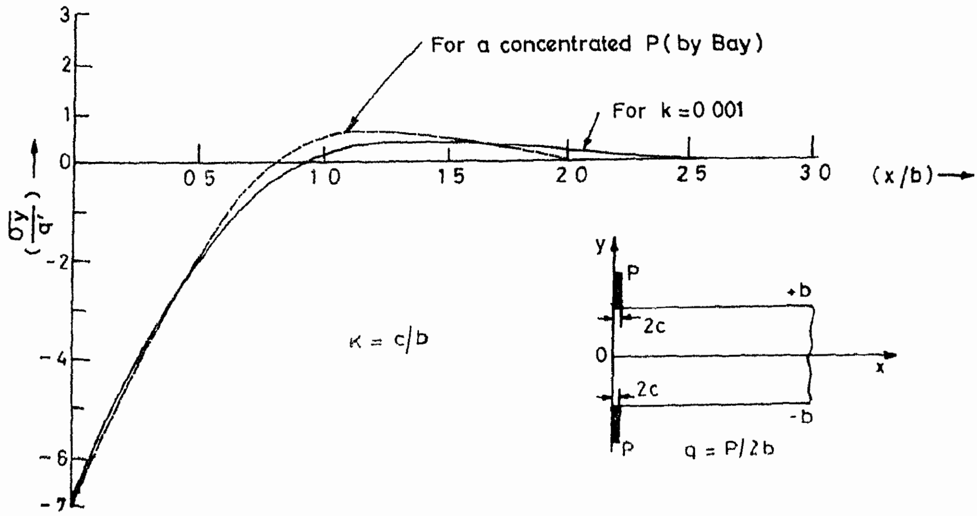

Fig 11 $\left(\frac{\sigma_y}{q'}\right)$ distribution along $(y/b) = 0.$

value of this tensile stress is not greater than 0 04 q in any case This type of distribution can be effectively combined with that due to normal compressive loadings on the transverse edge of a semi-infinite strip to achieve a zone free of tensile stresses near the end In case of prestressed post-tensioned beams because of longitudinal prestress, transverse tensile stresses are developed in the end block[2]. If compressive loading can be applied in the anchorage zone in the transverse direction, these tensile stresses can be nullified or reduced as it can be seen by combining the transverse stress distributions due to both the loadings.

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