Jour. Ind. Inst. Sc. 61 (B), June 1979, pp. 89–102
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Steady flow of a thermo-viscous fluid through straight tubes

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Received on February 10, 1979; Revised on April 28, 1979

Abstract

In this paper, the steady flow of a second order incompressible thermo-viscous fluid through straight tubes is studied. It is noticed that, as in the case of visco-elastic fluids, a purely rectilinear flow of thermo-viscous fluids in straight tubes, under the influence of a constant pressure gradient, is not sustainable. There exists a secondary flow in the tube cross-sections perpendicular to its axis. The flow through elliptic tube is illustrated as an example from which the flow through a circular pipe has been deduced as a special case.

Key words : Thermo-viscous fluids, heat flux bivector, thermal bigradient vector, secondary flow.

1. Introduction

Koh and Eringin[1] introduced the concept of thermo-viscous fluids which reflects the interaction between thermal and mechanical responses in fluids in motion due to external influences. For such a class of fluids, the stress-tensor t and heat flux bivector h are postulated to be polynomial functions of the kinematic tensors, *viz.*, the rate of deformation tensor d :

$$d_{ij} = (u_{i,j} + u_{j,i})/2 \tag{1.1}$$

and thermal gradient bivector b :

$$b_{ij} = \epsilon_{ijk}\theta_{,k} \tag{1.2}$$

where u_i is the ith component of velocity and θ is the temperature field.

A second order theory of thermo-viscous fluids is characterized by

$$t = \alpha_1 I + \alpha_3 d + \alpha_5 d^2 + \alpha_6 b^2 + \alpha_8 (db - bd) \tag{1.3}$$

$$h = \beta_1 b + \beta_3 (bd + db) \tag{1.4}$$

with the constitutive parameters α_i, β_i being polynomials in the invariants of d and b in which the coefficients depend on density and temperature only. When the fluid is Stokesian (*i.e.*, when the stress tensor depends only on the rate of deformation tensor), and Fourier-heat-conducting (*i.e.*, when the heat flux bivector depends only on the temperature gradient-vector), the constitutive coefficients α_1 and α_3 may be identified as the fluid pressure and coefficient of viscosity respectively and α_5 as that of cross-viscosity.

The flow of incompressible thermo-viscous fluids satisfies the usual conservation equations.

Equation of Continuity

$$v_{i,\,i} = 0. \tag{1.5}$$

Equation of Momentum

$$\rho\left[\frac{\partial v_i}{\partial t} + v_k v_{i,\,k}\right] = \rho F_k + t_{ji,\,i} \tag{1.6}$$

and Energy equation

$$\rho c\dot{\theta} = t_{ij}d_{ij} - q_{i,\,i} + \rho\nu \tag{1.7}$$

where

F_k = kth component of external force per unit mass,

c = specific heat;

ν = energy source per unit mass and

q_i = ith component of heat flux bivector.

$$q_i = \epsilon_{ijk}h_{jk}/2. \tag{1.8}$$

Solving a specific boundary value problem implies solving (1.5), (1.6) and (1.7) together with the constitutive equations (1.3) and (1.4), satisfying the appropriate boundary conditions such as the no-slip conditions. These equations are, apart from being nonlinear in character, coupled and may not be that easy to realize an exact solution for a specific boundary value problem.

In this communication, we adopt the four-step recursive approach proposed by Langlois and Rivlin[4] to obtain an approximate solution for the equations concerned to investigate the effect of non-Newtonian, non-Fourier terms in (1.3) and (1.4). The fluid is assumed to be in a state of steady flow within the volume occupied by the fluid and the boundaries are supposed to be rigid and at rest or moving in a specified manner. It is further assumed that there are no external forces and no heat source within the flow region. The four steps of the recursive procedure are detailed as follows.

Step I : A solution ($v_i^{(1)}$, $\alpha_1^{(1)}$, $\theta^{(1)}$) of the equations of motion and energy for an incompressible Newtonian-viscous and Fourier-heat conducting fluid in the absence of body forces and heat sources is obtained subject to suitable boundary conditions.

Step II : The solution ($v_i^{(1)}$, $\alpha_1^{(1)}$, $\theta^{(1)}$) obtained in step I is introduced into the equations of motion and energy of the incompressible thermo-viscous fluid described above, *i.e.*, equations (1.6) and (1.7) and the additional body force F_k and heat source $\rho\nu$ required to support this solution are calculated.

Step III : A solution ($v_i^{(2)}$, $\alpha_1^{(2)}$, $\theta^{(2)}$) of the equations of motion and energy for an incompressible Newtonian-Fourier heat conducting fluid, under the influence of body force F_k and heat source $\rho\nu$ computed in step II, is obtained subject to the homogeneous boundary conditions such as zero velocity/zero flux/zero temperature on the boundary Γ.

Step IV : Then ($\boldsymbol{v}_i = v_i^{(1)} - \boldsymbol{v}_i^{(2)}$, $\alpha_1 = \alpha_1^{(1)} - \alpha_1^{(2)}$, $\theta = \theta^{(1)} - \theta^{(2)}$) is an approximate solution for the equations of motion of a very slow motion of an incompressible, slightly thermo-viscous fluids, subject to the same boundary conditions used in step I.

Following the procedure sketched above, we examine hereunder, the steady flow of a second-order incompressible thermo-viscous fluid in a long straight non-circular pipe under the action of constant pressure gradient. The pipe wall is assumed to be fixed and maintained at a constant temperature θ_1. Let the axis of Z be chosen along the pipe line and the pipe represented by

$$\Gamma : \phi(x, y) = 0. \tag{1.9}$$

The pressure gradient down the tube axes is

$$\frac{\partial \alpha_1}{\partial z} = c_1. \tag{1.10}$$

2. Equation of motion and energy in different steps

Step I: The equation of motion and energy at this stage are same as those of Newtonian-viscous fluid and therefore purely rectilinear flow is sustainable. If $(0, 0, w^{(1)}(x, y))$ is the velocity and $\theta^{(1)}(x, y)$ is the temperature, then we have

$$\nabla^2 w^{(1)} = c_1/2\alpha_3. \tag{2.1}$$

and

$$\nabla^2 \theta^{(1)} = -\frac{\rho c\, c_2}{\beta_1} w^{(1)} \tag{2.2}$$

together with the boundary conditions

$$\left.\begin{array}{l} w^{(1)} = 0 \\ \theta^{(1)} = \theta_1 \end{array}\right\} \text{ on } \Gamma \tag{2.3}$$

Step II : $w^{(1)}$, $\theta^{(1)}$ realized above are substituted in the equations of motion and energy to obtain the body force F and heat source ν to support the flow, we thus obtain

$$\begin{aligned} \rho F_x = -\Big[& a_5 \left(\frac{2c_1}{a_3} w_x^{(1)} + w_x^{(1)} w_x^{(1)} + w_y^{(1)} w_{xy}^{(1)}\right)\Big/ 4 \\ & + a_6 \left(\theta_y^{(1)} \theta_{xy}^{(1)} - \theta_x^{(1)} \theta_{yy}^{(1)}\right) \\ & + a_8 \left(\frac{2c_1}{a_3} \theta_y^{(1)} + \theta_y^{(1)} w_{xx}^{(1)} + w_x^{(1)} \theta_{xy}^{(1)}\right. \\ & \left. + w_y^{(1)} \theta_{yy}^{(1)} - \theta_x^{(1)} w_{xy}^{(1)}\right)\Big/ 2 \Big] \end{aligned} \tag{2.4}$$

$$\begin{aligned} \rho F_y = -\Big[& a_5 \left(\frac{2c_1}{a_3} w_y^{(1)} + w_y^{(1)} w_{yy}^{(1)} + w_x^{(1)} w_{(1)}^{xy}\right)\Big/ 4 \\ & + a_6 \left(\theta_x^{(1)} \theta_{xy}^{(1)} - \theta_y^{(1)} \theta_{xx}^{(1)}\right) \\ & + a_8 \left(\frac{-2c_1}{a_3} \theta_x^{(1)} - \theta_x^{(1)} w_{yy}^{(1)} - w_y^{(1)} \theta_{xy}^{(1)}\right. \\ & \left. - w_x^{(1)} \theta_{xx}^{(1)} + \theta_y^{(1)} w_{xy}^{(1)}\right)\Big/ 2 \Big] \end{aligned} \tag{2.5}$$

$$F_z = -\left[c_1 + \frac{\rho c\, c_2^2\, a_6 w^{(1)}}{\beta_1} \right] \tag{2.6}$$

and heat source

$$\begin{aligned} \rho \nu = & -\frac{\beta_3 c_1 c_2}{a_3} - a_3 \left(w_x^{(1)^2} + w_y^{(1)^2}\right)/2 \\ & + a_6 c_2 \left(w_x^{(1)} \theta_x^{(1)} + w_y^{(1)} \theta_y^{(1)}\right) \end{aligned} \tag{2.7}$$

where $c_2 = \partial\theta/\partial x$ is the temperature gradient and is assumed to be constant.

Step III : The body force and heat source obtained above are substituted in the equations of motion and energy for a Newtonian viscous–Fourier heat conducting fluid. If $(u^{(2)}, v^{(2)}, w^{(2)})$ are the velocity components and $\theta^{(2)}(x, y)$ is the temperature field, these equations reduce to

$$0 = \frac{\partial a_1^{(2)}}{\partial x} + \frac{a_8}{2} \nabla^2 u^{(2)} + \rho F_x \tag{2.8}$$

$$0 = \frac{\partial a_1^{(2)}}{\partial y} + \frac{a_8}{2} \nabla^2 v^2 + \rho F_y \tag{2.9}$$

$$0 = \frac{\alpha_3}{2} \nabla^2 w^{(2)} + \rho \Gamma_* + c_1 \tag{2.10}$$

and

$$\rho c c_2 w^{(2)} = - \beta_1 \nabla^2 \theta^{(2)} - \frac{\beta_3 c_1 c_2}{\alpha_3} \tag{2.11}$$

with boundary conditions

$$u^{(2)} = 0, v^{(2)} = 0, w^{(2)} = 0, \text{ and } \theta^{(2)} = 0 \text{ on } \Gamma. \tag{2.12}$$

Introducing $\psi^{(2)}$, the stream function, by

$$u^{(2)} = - \frac{\partial \psi^{(2)}}{\partial y}, \quad v^{(2)} = \frac{\partial \psi^{(2)}}{\partial x}$$

so as to satisfy the continuity equation. Equations (2.8) and (2.9) yield after eliminating $\alpha^{(2)}$

$$\nabla^H \psi^{(2)} = \frac{2\rho c c_2}{\alpha_3 \beta_1} \left[\alpha_6 \frac{\partial (w^{(1)}, \theta^{(1)})}{\partial (x, y)} + \alpha_8 \left(\frac{4c_1}{\alpha_3} w^{(1)} + w_x^{(1)^2} + w_y^{(1)^2} \right) \Big/ 2 \right] \tag{2.13}$$

with

$$\psi_x^{(2)} = 0 = \psi_y^{(2)} \text{ on } \Gamma. \tag{2.14}$$

Step IV : An approximate solution for the flow through pipes can now be given by

The velocity field :

$$\begin{aligned} u &= u^{(1)} - u^{(2)} = - u^{(2)} \\ v &= v^{(1)} - v^{(2)} = - v^{(2)} \\ w &= w^{(1)} - w^{(2)} \end{aligned} \tag{2.15}$$

pressure

$$\alpha_1 = \alpha_1^{(1)} - \alpha_1^{(2)} \tag{2.16}$$

and the temperature field

$$\theta = \theta^{(1)} - \theta^{(2)}. \tag{2.17}$$

Thus a purely rectilinear flow of thermo-viscous fluids in straight tubes is not sustainable. The flow in the tube is composed of a rectilinear velocity $w = w^{(1)} - w^{(2)}$ down the tube axis over which is superposed a secondary flow characterized by the stream function $\psi^{(2)}$ obtained from the equation (2.13) in planes perpendicular to tube axis.

3. Flow through an elliptic tube

Let the cross-section Γ of the tube is given by

$$\phi(x, y) = x^2/a^2 + y^2/b^2 - 1 = 0. \tag{3.1}$$

Introducing the non-dimensional variables

$$x = aX, \ y = aY \text{ and } \sigma = b/a$$

(3.1) can be written as

$$\sigma^2 X^2 + Y^2 - \sigma^2 = 0. \tag{3.2}$$

The solution for eqns. (2.1) and (2.2) together with boundary conditions in (2.3), may be obtained as

$$w^{(1)} = \frac{c_1 a^2}{\alpha_3} \cdot \frac{1}{1+\sigma^2} (\sigma^2 X^2 + Y^2 - \sigma^2) \tag{3.3}$$

$$\theta^{(1)} = \theta_1 + \left(\frac{c c_1 c_2 a^2 \rho}{12 \alpha_3 \beta_1}\right) \frac{1}{(1+\sigma^2) f} [\sigma^2 (5+\sigma^2) (1+5\sigma^2) - \sigma^2 (1+\sigma^2)(5+\sigma^2) X^2 - (1+\sigma^2)(1+5\sigma^2) Y^2] [\sigma^2 X^2 + Y^2 - \sigma^2] \tag{3.4}$$

Substituting (3.3) and (3.4) in (2.4), (2.5), (2.6) and (2.7), we can obtain body force and heat source to sustain the velocity and temperature as

$$\begin{aligned} \rho F_x = & - \frac{c_1^2 \alpha_5 a}{\alpha_3^2} \frac{1+2\sigma^2}{(1+\sigma^2)^2} X \\ & + \frac{\alpha_8 \rho c c_1^2 c_2 a^3}{12 \alpha_3^2 \beta_1 (1+\sigma^2) f} [-8(1+2\sigma^2)(1+5\sigma^2) Y^3 \\ & - 24\sigma^2 (1+\sigma^2)(1+2\sigma^2) X^2 Y + 8\sigma^2 (1+5\sigma^2)(3+\sigma^2) Y] \\ & + \alpha_6 \left(\frac{\rho c c_1 c_2}{12 \alpha_3 \beta_1}\right)^2 \frac{a^5}{(1+\sigma^2)^2 f} [48\sigma^2 (1+\sigma^2)^3 (1+5\sigma^2) X Y^4 \\ & - 96\sigma^4 (1+5\sigma^2) f X Y^2 + 96\sigma^4 (1+\sigma^2)^2 (\sigma^4 + 10\sigma^2 + 1) X^2 Y^3 \\ & - 48\sigma^6 (1+\sigma^2)^3 (5+\sigma^2) X^5 \\ & - 16\sigma^6 (5+\sigma^2)(1+\sigma^2)(14\sigma^4 + 28\sigma^2 + 6) X^3 \\ & + 16(5+\sigma^2)(1+5\sigma^2)(1+3\sigma^2)(3+\sigma^2)\sigma^2 X] \end{aligned} \tag{3.5}$$

$$\begin{aligned} \rho F_y = & - \frac{\alpha_5 c_1^2 a (2+\sigma^2)}{\alpha_3^2 (1+\sigma^2)^2} Y \\ & + \frac{\rho c c_1^2 c_2 \alpha_8 a^3}{12 \alpha_3^2 \beta_1 f (1+\sigma^2)} [8\sigma^4 (1+2\sigma^2)(5+\sigma^2) X^3 \\ & - 4\sigma^4 (5+\sigma^2)(1+3\sigma^2) X + 24\sigma^2 (1+\sigma^2)(2+\sigma^2) X Y^2] \\ & + \alpha_6 \left(\frac{\rho c c_1 c_2}{12 \alpha_3 \beta_1}\right)^2 \frac{a^5}{(1+\sigma^2)^2 f} [48\sigma^6 (1+\sigma^2)^3 (5+\sigma^2) X^2 Y \end{aligned}$$

$$+ 96\sigma^4(1+\sigma^2)^2\ (1+10\sigma^2+\sigma^4)X^2Y^3 - 96\sigma^6 f\ (5+\sigma^2)\ X^2 Y$$
$$+ 48\sigma^2(1+5\sigma^2)\ (1+\sigma^2)^3\ Y^5$$
$$- 16\sigma^4(6\sigma^4+28\sigma^2+14)\ (1+\sigma^2)\ Y^3$$
$$+ 16\sigma^6(5+\sigma^2)(1+5\sigma^2)\ (1+3\sigma^2)\ (3+\sigma^2)\ Y]. \tag{3.6}$$

$$\rho F_s = -\frac{\rho c c_1 c_2^2\, \alpha_6 a^2}{\alpha_3\beta_1}\left[\frac{\beta_1\alpha_3}{\rho c c_2^2\,\alpha_6 a^2} + \frac{\sigma^2 X^2 + Y^2 - \sigma^2}{\sigma^2+1}\right] \tag{3.7}$$

and the heat source is

$$\rho v = -\frac{\beta_3 c_1 c_2}{\alpha_3} - \frac{2c_1^2\, a^2}{\alpha_3(1+\sigma^2)^2}(\sigma^4 X^2 + Y^2)$$
$$- \frac{2\rho c c_2^2\, c_1^2\, a^4}{3\alpha_3^2\,\beta_1}\ [-\sigma^6(5+\sigma^2)\ (1+\sigma^2)X^4 - 3\sigma^2(1+\sigma^2)^3 X^2Y^2$$
$$-(\sigma^2+1)\ (1+5\sigma^2)\ Y^4 + (1+3\sigma^2)\ (5+\sigma^2)\sigma^6 X^2$$
$$+(3+\sigma^2)\ (1+5\sigma^2)\ \sigma^2 Y^2]/f(1+\sigma^2)^2 \tag{3.8}$$

with $f = (1+\sigma^2)\ (1+6\sigma^2+\sigma^4)$.

Using these in equations of motion (2.8) and (2.9) and then eliminating pressure, the equations of motion and energy reduce to

$$\nabla^4\psi^{(2)} = \frac{2\rho c c_1^2\, c_2}{\alpha_3^2\,\beta_1(a^2+b^2)^2}\Big[-\frac{16\rho c c_2\alpha_6}{\beta_1}\, a^2 b^2\ (a^2-b^2)\ (b^2x^2+a^2y^2-a^2b^2)\, xy$$
$$+ 2\alpha_8\,\{b^2(a^2+2b^2)\,x^2 + a^2(2a^2+b^2)\,y^2 - a^2b^2(a^2+b^2)\}\Big] \tag{3.9}$$

$$\nabla^2 w^{(2)} = \frac{2\alpha_6\, c_2^2\, c_1\rho c}{\alpha_3^2\,\beta_1(a^2+b^2)}(b^2x^2+a^2y^2-a^2b^2) \tag{3.10}$$

and

$$\nabla^2\theta^{(2)} = -\frac{\beta_3 c_1 c_2}{\alpha_3\beta_1} - \frac{\rho c c_2}{\beta_1}w^{(2)} \tag{3.11}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and boundary conditions given in (2.12) and (2.14).

Equation (3.9) together with (2.14) yields the solution

$$\psi^{(2)} = K\psi_1^{(2)} + \psi_2^{(2)}. \tag{3.12}$$

where

$$\psi_1^{(2)} = (\sigma^2 X^2 + Y^2 - \sigma^2)^2 (-s_0 + s_1 X^2 + s_2 Y^2)/g \tag{3.13}$$

and

$$\psi_2^{(2)} = \sigma^2 (1 - \sigma^2)(\sigma^2 X^2 + Y^2 - \sigma^2)^2 \ (s_3 - s_4 X^2 - s_5 Y^2) XY/h \tag{3.14}$$

with

$$s_0 = \sigma^2 (495\sigma^{14} + 2487\sigma^{12} + 12031\sigma^{10} + 9339\sigma^8 + 4345\sigma^6 + 1701\sigma^4$$

$$+ 289\sigma^2 + 33)/(3\sigma^4 + 2\sigma^2 + 3)(15\sigma^4 + 4\sigma^2 + 1)$$

$$s_1 = \sigma^2 (30\sigma^{10} + 113\sigma^8 + 450\sigma^6 + 284\sigma^4 + 72\sigma^2 + 11)/(15\sigma^4 + 4\sigma^2 + 1)$$

$$s_2 = (11\sigma^6 + 28\sigma^4 + 7\sigma^2 + 2)$$

$$s_3 = \sigma^2 (175\sigma^{12} + 728\sigma^{10} + 2071\sigma^8 + 1286\sigma^6 + 677\sigma^4 + 184\sigma^2 + 25)/(7\sigma^4$$

$$+ 4\sigma^2 + 1) \ (5\sigma^4 + 6\sigma^2 + 5)$$

$$s_4 = (5 + 22\sigma^2 + 44\sigma^4 + 14\sigma^6 + 7\sigma^8)/(7\sigma^4 + 4\sigma^2 + 1)$$

$$s_5 = 1 + 2\sigma^2 + 5\sigma^4$$

where

$$g = 150\sigma^8 + 60\sigma^6 + 234\sigma^4 + 60\sigma^2 + 15$$

$$h = 7\sigma^8 + 28\sigma^6 + 58\sigma^4 + 28\sigma^2 + 7$$

and $K = 5\alpha_8 \beta_1 / 8\rho c \, c_2 \alpha_6 a^6$.

The secondary flow in planes perpendicular to the tube length for a thermo-viscous fluid is thus composed of two components $\psi_1^{(2)}$ and $\psi_2^{(2)}$ which are illustrated in Figs. 1 and 2 (for $\sigma = 0.4$). The pattern $\psi_1^{(2)}$ (Fig. 1) is composed of the flow around a single vortex around the centre of ellipse. The flow pattern $\psi_2^{(2)}$ (Fig. 2) is rotational consisting of flow around four vortices in four quadrants symmetrically placed about the axes of ellipse. This flow is similar to the one noticed by Green and Rivlin[2] for a Riner–Rivlin fluid. It is noticed that for $K = 1.0$ the component $\psi_1^{(2)}$ dominates over $\psi_2^{(2)}$ and the resulting stream line pattern $\psi^{(2)}$ is much very close to that of $\psi_1^{(2)}$ but drifted towards the boundary (Fig. 1).

The axial velocity component $w^{(2)}$ can be noticed to be

$$w^{(2)} = \left(\frac{\alpha_6 c_2^2 c_1 \rho c a^4}{\alpha_8^2 \beta_1}\right) [-\sigma^2 (5 + \sigma^2)(1 + 5\sigma^2) + \sigma^2 (5 + \sigma^2)(1 + \sigma^2) X^2$$

$$+ (1 + \sigma^2)(1 + 5\sigma^2) Y^2] (\sigma^2 x^2 + Y^2 - \sigma^2)/f(1 + \sigma^2) \tag{3.15}$$

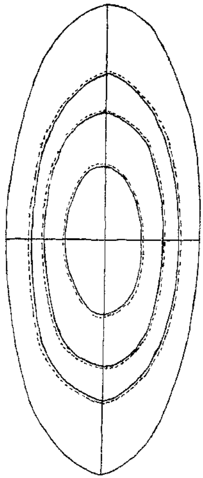

FIG. 1. Stream line pattern. — $\psi^2_{(1)}$ - - - $\psi^{(2)}$.

and the temperature field $\theta^{(2)}$ as

$$\theta^{(2)} = \alpha_6 \frac{c_1 c_2^3 \rho^2 c^2 a^6}{\alpha_3^2 \beta_1^2} [A_0 + B_1 X^2 + B_3 Y^2 - (D_1 X^4 + D_3 X^2 Y^2 + D_5 Y^4)] (\sigma^2 X^2 + Y^2 - \sigma^2) \quad (3.16)$$

where

$$D_1 = \sigma^4 (46 + 64\sigma^2 + 18\sigma^4 + \sigma^6)/f_1$$

$$D_3 = \sigma^2 (14 + 161\sigma^2 + 147\sigma^4 + 14\sigma^6)/f_1$$

$$D_5 = (1 + 19\sigma^2 + 78\sigma^4 + 61\sigma^6)/f_1$$

$$B_3 = [(2 + 21\sigma^2 + 63\sigma^4 + 19\sigma^6) + 4 (D_3 + 6D_5) f\sigma^2 (1 + \sigma^2) - 4 (D_3 + 6D_1) f\sigma^2 (\sigma^2 + 6) (\sigma^2 + 1)]/24f,$$

$$B_1 = [\sigma^4 (8 + 23\sigma^2 + 8\sigma^4 + \sigma^6) + 4 (D_3 + 6D_5) f\sigma^4 (1 + \sigma^2) - 4 (D_3 + 6D_1) f\sigma^2 (\sigma^2 + 6) (\sigma^2 + 1)]/24f$$

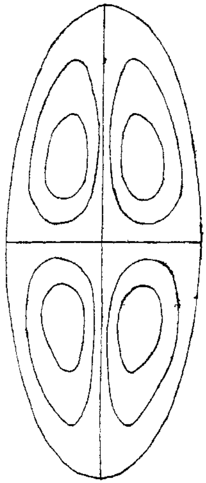

FIG. 2. Stream line pattern $\psi_2^{(2)}$.

and

$$A_0 = [\sigma^2 (\sigma^4 - \sigma^2 - 1) \ (D_3 + 6D_5)/6f - \sigma^4 (\sigma^2 + 5) \ (D_3 + 6D_1)/6f$$
$$+ \sigma^2 \{(\sigma^{10} + 8\sigma^8 + 42\sigma^6 + 71\sigma^4 + 21\sigma^2 + 2)$$
$$- 4\sigma^2 f (1 + 5\sigma^2) \ (\sigma^2 + 5)\}/24f^2 (1 + \sigma^2) - L/(1 + \sigma^2)]$$

with

$$L = \beta_3 a_3 \beta_1 / a_6 c_2^2 \rho^2 \, c^2 a^4$$

and

$$f_1 = 180f(1 + 15\sigma^2 + 15\sigma^4 + \sigma^6).$$

The expressions for $w^{(2)}$ and $\theta^{(2)}$ vanish when the non-Fourier coefficients a_6, β_3 vanish.

The Nussult Number which characterize the heat transfer coefficient on the boundary is computed as

$$Nu = \hat{n} \, . \, \nabla \theta \tag{3.17}$$

where $\hat{n}$ is the unit normal to the boundary

$$Nu = Nu_1 - Nu_2 \tag{3.18}$$

where

$$Nu_1 = [\sigma^2 (5 + \sigma^2)(1 + 5\sigma^2) - \sigma^2 (1 + \sigma^2)(5 + \sigma^2) X^2 - (1 + \sigma^2)(1 + 5\sigma^2) Y^2] \sqrt{\sigma^4 X^2 + Y^2}/6 (1 + \sigma^2) f \tag{3.19}$$

and

$$Nu_2 = 2 [A_0 + B_1 X^2 + B_3 Y^2 - (D_1 X^4 + D_3 X^2 Y^2 + D_5 Y^4)] \sqrt{\sigma^4 X^2 + Y^2} + 2(\sigma^2 X^2 + Y^2 - \sigma^2) [\sigma^2 (B_1 - 2D_1 X^2 - D_3 Y^2) X^2 + (B_3 - D_3 X^2 - 2D_5 Y^2) Y]/ \sqrt{\sigma^2 X^2 + Y^2}. \tag{3.20}$$

The numerical estimate of these parameters shows that $Nu_2 < Nu_1$ and Fig. 3 illustrate the variation of the Nussult Number on the boundary.

The mass transfer in the tube is given by

$$Q = \iint_A w\,dx\,dy = \iint_A (w^{(1)} - w^{(2)})\,dx\,dy = \frac{\pi}{2}\left[\frac{-3c_1 a^2}{\alpha_3} \frac{\sigma^3}{1 + \sigma^2} - 4 \cdot \frac{\alpha_6 c_2^2 c_1 \rho c a^4}{\alpha_3^2 \beta_1 (1 + \sigma^2)} \sigma^5 (5\sigma^4 + 26\sigma^2 + 1)\right] \tag{3.21}$$

The effect of α_6 is to reduce the flux of the fluid through any cross-section of the tube.

Velocity components $u^{(2)}$ and $v^{(2)}$ are obtained as

$$u^{(2)} = -(\sigma^2 X^2 + Y^2 - \sigma^2) \{2KY [2 (- s_0 + s_1 X^2 + s_2 Y^2) + s_2 (\sigma^2 X^2 + Y^2 - \sigma^2)] + X [(s_3 - s_4 X^2 - s_5 Y^2)(\sigma^2 X^2 + Y^2 - \sigma^2) - 2s_5 Y^2 (\sigma^2 X^2 + Y^2 - \sigma^2)]\} \tag{3.22}$$

$$v^{(2)} = (\sigma^2 X^2 + Y^2 - \sigma^2) \{2KX [2\sigma^2 (- s_0 + s_1 X^2 + s_2 Y^2) + s_1 (\sigma^2 X^2 + Y^2 - \sigma^2)] + Y [(s_3 - s_4 X^2 - s_5 Y^2) (5\sigma^2 X^2 + Y^2 - \sigma^2) - 2s_4 X^2 (\sigma^2 X^2 + Y^2 - \sigma^2)]\}. \tag{3.23}$$

4. Flow through a circular pipe

The case of circular tube can be realized as a special case by putting $\sigma = 1 \cdot 0$.
We notice that

$$w^{(1)} = -\frac{c_1 a^2}{2\alpha_3} (1 - X^2 - Y^2) \tag{4.1}$$

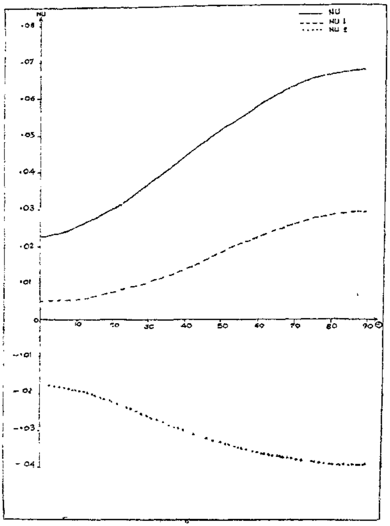

FIG. 3. *NU* vs. the eccentric angle θ.

and

$$\theta^{(1)} = \theta_1 + \frac{\rho c c_2 c_1 a^4}{32 \alpha_3 \beta_1} \; [4 (X^2 + Y^2) - (X^2 + Y^2)^3 - 3]. \tag{4.2}$$

The body force and heat source are given by

$$\rho F_x = - \Big[- \frac{3}{4} \frac{c_1^2 \alpha_5 a}{\alpha_3^2} X - 16 \alpha_6 \left(\frac{c_1 c_2 \rho c}{32 \alpha_3 \beta_1} \right)^2 a^5 (X^2 + Y^2 - 1) X + \frac{\alpha_8 c_1^2 c_2 \rho c a^3}{8 \alpha_3^2 \beta_1} \{4 - 3 (X^2 + Y^2)\} Y \Big] \tag{4.3}$$

$$\rho F_y = - \Big[- \frac{3}{4} \frac{c_1^2 \alpha_5 a}{\alpha_3^2} Y - 16 \alpha_6 \left(\frac{c_1 c_2 \rho c}{32 \alpha_3 \beta_1} \right)^2 a^5 (X^2 + Y^2 - 1) Y + \frac{\alpha_8 c_1^2 c_2 \rho c a^3}{8 \alpha_3^2 \beta_1} \{3(X^2 + Y^2) - 4\} X \Big] \tag{4.4}$$

$$F_z = -\left[c_1 + \frac{\rho c \alpha_6 c_2^2 c_1 a^2}{2\alpha_3 \beta_1}(X^2 + Y^2 - 1)\right] \tag{4.5}$$

and heat source

$$\rho r = -\left[\frac{\beta_3 c_1 c_2}{\alpha_3} + \frac{c_1^2 a^2}{2\alpha_3}(X^2 + Y^2) - \frac{\alpha_6 c_2^2 c_1^2 \rho c a^4}{8\alpha_3^2 \beta_1}(X^2 + Y^2)(2 - X^2 - Y^2)\right]. \tag{4.6}$$

The stream function is given by

$$\psi^{(2)} = Ca^6 (X^2 + Y^2 - 1)^2 (X^2 + Y^2 - 4)/192 \tag{4.7}$$

with $C = \alpha_6 c_1^2 c_2 \rho c/\alpha_3^3 \beta_1$.

The velocity components are

$$u^{(2)} = - Ca^5 Y(X^2 + Y^2 - 1)(X^2 + Y^2 - 3)/32 \tag{4.8}$$

$$v^{(2)} = Ca^5 X(X^2 + Y^2 - 1)(X^2 + Y^2 - 3)/32. \tag{4.9}$$

The axial velocity component is

$$w^{(2)} = \frac{\alpha_6 c_2^2 c_1 c a^4}{16 \alpha_3^2 \beta_1}[3-4(X^2 + Y^2) + (X^2 + Y^2)^2] \tag{4.10}$$

and the temperature field is deduced as

$$\theta^{(2)} = -\frac{\beta_3 c_1 c_2 a^4}{4\alpha_3 \beta_1}(X^2 + Y^2 - 1) + \frac{\alpha_6 c_1 c_2^3 c^2 \rho^2 a^6}{576\alpha_3^2 \beta_1}(X^2 + Y^2)\{-27 + 9(X^2 + Y^2) - (X^2 + Y^2)^2\} \tag{4.11}$$

It is interesting to note that there is marked departure between the flows of a thermo-viscous fluid as compared to that of visco-elastic fluid in circular tubes.

It is noticed by Ericksen[3] that a purely rectilinear flow down a circular tube can be maintained under the influence of constant axial pressure gradient. This would not be the case for a thermo-viscous fluid flow in a circular tube. This flow given by $w^{(2)}$ and $\phi^{(2)}$ is therefore helical in nature.

Another point of contrast between the thermo-viscous and visco-elastic fluids has been noticed by the authors[7]. In a Couette flow of a thermo-viscous fluid, a temperature gradient depending on the velocity of the moving plate is generated within the flow region whereas such a phenomenon does not happen for the classical viscous and visco-elastic fluids.

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