On approximate strong and approximate uniform differentiability

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Abstract

In this paper we introduce the definitions of approximate strong and approximate uniform differentiability and prove that a function having finite approximate derivative on an interval I is approximately uniformly differentiable at a point if and only if it is approximately strongly differentiable there. We also introduce the definition of essentially AC [BV] functions and prove that a measurable BVG function on a set with finite measure is essentially AC.

1. Introduction

Let f be a real-valued function on an open interval I. Lahiri[1] introduced the following definition.

Definition 1.1: Let $f'(x)$ exist finitely at each point of I. A point $a \in I$ is said to be a point of uniform differentiability of f if for every $\varepsilon > 0$ there is a neighbourhood $\triangle_a(\delta) = (a - \delta, a + \delta) \subset I$ and a positive number P such that

$$\left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

for all $x \in \triangle_a(\delta)$ and $x + h \in I$ whenever $0 < |h| \leq P$. Otherwise a is said to be a point of non-uniform differentiability of f.

Lahiri proved the following results.

Theorem 1.1: Let $f'(x)$ exist finitely at each point of I. A point $a \in I$ is a point of uniform differentiability of f iff f' is continuous at a.

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Theorem 1.2: Let $f'(x)$ exist finitely at each point of I. Then the set of the points of non-uniform differentiability of f in I is of first category and is an F_σ.

Esser and Shisha[1] introduced the following definition.

Definition 1.2: f is said to be strongly differentiable at $a \in I$ if the double limit

$$\lim_{\substack{(x, y) \to (a, a) \\ x \neq y}} \frac{f(x) - f(y)}{x - y} \quad \text{exists finitely.}$$

This limit whenever exists is denoted by $f^*(a)$ and is called the strong derivative of f at a.

Esser and Shisha proved the following result.

Theorem 1.3: Let $f'(x)$ exist finitely at each point of I. Then f is strongly differentiable at $a \in I$ iff f' is continuous at a.

From Theorems 1.1 and 1.3 we immediately derive the following result.

Theorem 1.4: Let $f'(x)$ exist finitely on I. A point $a \in I$ is a point of uniform differentiability of f iff f is strongly differentiable at a.

In Section 2, we prove some results on strong and uniform differentiability. In Section 3, we introduce the definitions of approximate strong and approximate uniform differentiability and extend the above results in the light of these new definitions and prove some other results. In Section 4, we introduce the definition of essentially AC [BV] functions and prove, among other results, that if f is BVG and measurable on a set E with $mE < +\infty$, then f is essentially AC on E.

2. Results on strong and uniform differentiability

Definition 2.1: Let f and g be two real-valued functions on $[a, b]$. Suppose that for every $a \in (a, b)$ there is a set $S_a \subset (a, b)$ having a as two-sided limiting point such that

$$\lim_{\substack{x \to a \\ x \in S_a}} \frac{f(x) - f(a)}{x - a} \quad \text{exists and equals } g(a).$$

At the points a and b necessary modifications are made. In this case we call g a derived function of f on $[a, b]$.

Note 2.1: If the approximate derivative $f'_{ap}(x)$ exists finitely on $[a, b]$, then f'_{ap} is a derived function of f on $[a, b]$.

Theorem 2.1: Let f be a real-valued bounded function on $[a, b]$ and let f attain its bounds on $[a, b]$. If f possesses a derived function g on (a, b) and $f(a) = f(b)$, then there is a point c in (a, b) such that $g(c) = 0$.

Proof: If $f(x) = f(a) = f(b)$ for all $x \in [a, b]$, then $g(x) = 0$ on (a, b). Suppose that there is a point ξ in (a, b) with $f(\xi) > f(a) = f(b)$. Denote by M the l.u.b. of f on $[a, b]$. Then there is a point c in $[a, b]$ such that $M = f(c)$. Since $M > f(a) = f(b)$ we have $a < c < b$ and

$$f(x) - f(c) \leqslant 0 \text{ for all } x \in [a, b]. \tag{2.1}$$

For any x in (c, b) we get

$$\frac{f(x) - f(c)}{x - c} \leqslant 0.$$

Letting $x \to c$ over the set S_a (see def. 2.1), we obtain

$$g(c) \leqslant 0. \tag{2.2}$$

Again for any x in (a, c) we get from (2.1)

$$\frac{f(x) - f(c)}{x - c} \geqslant 0.$$

Letting $x \to c$ over the set S_a we have

$$g(c) \geqslant 0. \tag{2.3}$$

Combining (2.2) and (2.3) we obtain $g(c) = 0$.

If there is a point η in (a, b) with $f(\eta) < f(a) = f(b)$, then proceeding as above and using the property of g.l.b. of f on $[a, b]$ we deduce that $g(c) = 0$ for some point c in (a, b). This completes the proof.

Corollary 2.1.1: Let f be continuous on $[a, b]$ and possess a derived function g on (a, b). Then there is a point c in (a, b) such that

$$f(b) - f(a) = (b - a) g(c).$$

Corollary 2.1.2: Let f be continuous on $[a, b]$ and let $f'_{ap}(x)$ exist finitely at each point of (a, b). Then there is a point c in (a, b) such that

$$f(b) - f(a) = (b - a) f'_{ap}(c).$$

Theorem 2.2: Let f be continuous on the open interval I and let f possess a derived function g on I. Then f is strongly differentiable at $a \in I$ iff g is continuous at a.

Proof: First suppose that g is continuous at a. Choose any $\varepsilon > 0$. Then there is a $\delta > 0$ such that $(a - \delta, a + \delta) \subset I$ and

$$|g(x) - g(a)| < \varepsilon \text{ for all } x \in (a - \delta, a + \delta). \tag{2.4}$$

Take any two points x, y $(x \neq y)$ in $(a - \delta, a + \delta)$. By Corollary 2.2.1, there is a point ξ lying between x and y such that

$$f(x) - f(y) = (x - y) g(\xi).$$

Clearly $\xi \in (a - \delta, a + \delta)$. So using (2.4) we have

$$\left| \frac{f(x) - f(y)}{x - y} - g(a) \right| = |g(\xi) - g(a)| < \varepsilon.$$

This gives that $f^*(a)$ exists and $f^*(a) = g(a)$.

Next, let f be strongly differentiable at a. Then $f'(a)$ exists and $f'(a) = g(a) = f^*(a)$. Choose any $\varepsilon > 0$. There is a $\delta > 0$ such that $(a - \delta, a + \delta) \subset I$ and

$$\left| \frac{f(x) - f(y)}{x - y} - g(a) \right| < \varepsilon \tag{2.5}$$

for all x, y $(x \neq y)$ in $(a - \delta, a + \delta)$. Keeping x fixed and letting $y \to x$ over set S, (see def. 2.1) in (2.5) we get

$$|g(x) - g(a)| \leqslant \varepsilon.$$

This gives that g is continuous at a.

Corollary 2.2.1: Let f be continuous on $[a, b]$ and possess a derived function g on $[a, b]$. If g is continuous on $[a, b]$, then $f'(x)$ exists on $[a, b]$ and $f'(x) = g(x)$ on $[a, b]$.

Corollary 2.2.2: Let f be continuous on $[a, b]$ and let $f'_{ap}(x)$ exist finitely at each point of $[a, b]$. Then f is strongly differentiable at each point of continuity of f'_{ap}.

Definition 2.2: Let f be a real-valued function on the set E and let $a \in E$ and be a limiting point of E. If the ratio

$$\frac{f(x) - f(a)}{x - a}$$

tends to a limit (finite or infinite) as x tends to a over the set E, we denote this limit by $f'_E(a)$ and call it the derivative of f at a relative to the set E.

Definition 2.3: Let f be a real-valued function on the set E which is dense in itself and let $f'_E(x)$ exist finitely at each point of E. f is said to be uniformly differentiable on E if for any $\varepsilon > 0$ there is a positive number δ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'_E(x) \right| < \varepsilon$$

for all x, y $(x \neq y)$ in E whenever $|x - y| < \delta$.

Theorem 2.3: Let f be a real-valued function measurable on the set E which is dense in itself and $mE < + \infty$ and let $f'_E(x)$ exist finitely at each point of E. Then given

any $\varepsilon > 0$ there is a perfect set $C \subset E$ such that $m(E - c) < \varepsilon$ and f is uniformly differentiable on C.

Proof: Clearly f is continuous on E. By Theorem 4.3 ([5], Ch. IV, p. 113) f'_E is measurable on E. Let ε be any given positive number. Then by Lusin's Theorem ([5], Theorem 7.1, Ch. III, p. 72) there is a closed set $F \subset E$ such that $m(E - F) < \frac{1}{2}\varepsilon$ and f'_E is continuous on F.

For any two positive integers K and n denote by $E_n(K)$ the set of all points $x \in F$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'_E(x) \right| \leqslant 2^{-K} \tag{2.6}$$

whenever $y \in E$, $y \neq x$ and $|y - x| \leqslant 1/n$. Then clearly for each K,

$$E_1(K) \subset E_2(K) \subset E_3(K)\ldots \quad \text{and} \quad F = \cup_{n=1}^{\infty} E_n(K).$$

Let $x \in \bar{E}_n(K)$ and $y \in E$, $y \neq x$ and $|y - x| \leqslant 1/2n$. If $x \in E_n(K)$, then clearly (2.6) holds. Suppose that $x \notin E_n(K)$. Then we can choose a sequence $\{x_\nu\}$ of points in $E_n(K)$ such that $x_\nu \to x$ as $\nu \to +\infty$. We may suppose that $|x - x_\nu| \leqslant 1/2n$ and $x_\nu \neq y$ for $\nu = 1, 2, 3, \ldots$. Then $|y - x_\nu| \leqslant |y - x| + |x - x_\nu| \leqslant 1/n$.

So

$$\left| \frac{f(y) - f(x_\nu)}{y - x_\nu} - f'_E(x_\nu) \right| \leqslant 2^{-K} \ (\nu = 1, 2, 3, \ldots).$$

Since f and f'_E are continuous on F, letting $\nu \to +\infty$ we see that (2.6) holds. Thus

$$\left| \frac{f(y) - f(x)}{y - x} - f'_E(x) \right| \leqslant 2^{-K} \tag{2.7}$$

for $x \in \bar{E}_n(K)$ and $y \in E$, $y \neq x$ whenever $|x - y| \leqslant 1/2n$. We have for each K

$$\bar{E}_1(K) \subset \bar{E}_2(K) \subset \bar{E}_3(K) \subset \ldots \quad \text{and} \quad F = \cup_{n=1}^{\infty} \bar{E}_n(K).$$

So, for each K, there is a positive integer n_K such that

$$m\left(F - \bar{E}_{n_K}(K)\right) < \varepsilon/2^{K+1}.$$

Take $B_K = \bar{E}_{n_K}(K)$ and $B = \cap_{K=1}^{\infty} B_K$.

Then B is closed and $m(F - B) \leqslant \sum_{k=1}^{\infty} m(F - B_K) < \frac{1}{2}\varepsilon$. Denote by C the set of all condensation points of B. Then C is a perfect set, $C \subset B \subset F \subset E$ and

$$m(E - c) \leqslant m(E - F) + m(F - B) + m(B - c) < \varepsilon.$$

Let η be any positive number. Choose positive integer K such that $2^{-K} < \eta$.

Take $\delta = 1/2n_K$. Let $x, y \in C$, $y \neq x$ and $|x - y| < \delta$. Then $x, y \in \bar{E}_{n_K}(k)$ and $|x - y| < 1/2n_K$. So by (2.7)

$$\left| \frac{f(y) - f(x)}{y - x} - f'_E(x) \right| \leqslant 2^{-K} < \eta.$$

This gives that f is uniformly differentiable on C.

Theorem 2.4 : Let f be a real-valued function measurable on the set E which is dense in itself and $mE < +\infty$. Suppose that $f'_E(x)$ exists finitely at each point of E. Then given any $\varepsilon > 0$, there is a perfect set $C \subset E$ with $m(E - C) < \varepsilon$ and a real-valued function g on the real line having the following properties: (i) g possesses continuous derivative on the real line, (ii) $g(x) = f(x)$ and $g'(x) = f'_E(x)$ for all $x \in C$.

Proof : Choose any $\varepsilon > 0$. Then by Theorem 2.3 there exists a bounded perfect set $C \subset E$ with $m(E - C) < \varepsilon$ such that f is uniformly differentiable on C. Clearly f and f'_E are continuous on C.

Denote by a and b the g.l.b. and l.u.b. of the set C and let $G = [a, b] - C$. Then G is an open set and so can be expressed in the form $G = \cup_n (a_n, b_n)$, where the open intervals $(a_1, b_1), (a_2, b_2), \ldots$ are pairwise disjoint.

We define the function g_n on $[a_n, b_n]$ by

$$g_n(x) = A_n(x - a_n)^3 + B_n(x - a_n)^2 + (x - a_n)f'_E(a_n) + f(a_n),$$

where A_n and B_n are constants. We choose them such that

$$g_n(b_n) = f(b_n), \quad g'_n(b_n) = f'_E(b_n). \quad \text{Then}$$

$$A_n(b_n - a_n)^2 = \{f'_E(b_n) - f'_E(a_n)\} - 2\left\{ \frac{f(b_n) - f(a_n)}{b_n - a_n} - f'_E(a_n) \right\},$$

$$B_n(b_n - a_n) = 3\left\{ \frac{f(b_n) - f(a_n)}{b_n - a_n} - f'_E(a_n) \right\} - \{f'_E(b_n) - f'_E(a_n)\}.$$

We now define the functions g and h on the real line as follows:

$g(x) = f(x), \quad h(x) = f'_E(x)$ for $x \in C$,

$g(x) = g_n(x), \quad h(x) = g'_n(x)$ for $a_n < x < b_n$,

$g(x) = (x - a)f'_E(a) + f(a), \quad h(x) = f'_E(a)$ for $x < a$,

$g(x) = (x - b)f'_E(b) + f(b), \quad h(x) = f'_E(b)$ for $x > b$.

Clearly g and h are continuous at each point of the set $A = G \cup (-\infty, a) \cup (b, \infty)$; $g'(x)$ exists at each point of A and $g'(x) = h(x)$. If $a \in C$, then

$$\lim_{\substack{x \to a \\ x \in C}} \frac{g(x) - g(d)}{x - a} = \lim_{\substack{x \to a \\ x \in C}} \frac{f(x) - f(a)}{x - a} = f'_E(a) = h(a).$$

This gives that h is a derived function of g on $(-\infty, \infty)$.

We now show that g and h are continuous at each point of C.

Case I. Let $G = \cup_{n=1}^{\infty} (a_n, b_n)$.

Choose any $\eta > 0$. Since f and f'_E are uniformly continuous on C and f is uniformly differentiable on C, there is a δ with $0 < \delta < \min \{\eta, 1\}$ such that

$$\begin{cases} |f(x) - f(y)| < \eta, \ |f'_E(x) - f'_E(y)| < \eta, \\ \left| \dfrac{f(y) - f(x)}{y - x} - f'_E(x) \right| < \eta, \end{cases} \tag{2.8}$$

for all x, y $(x \neq y)$ in C whenever $|x - y| < \delta$.

We choose a positive integer N such that

$$\sum_{n=N+1}^{\infty} (b_n - a_n) > \tfrac{1}{2} \delta. \tag{2.9}$$

Let $a \in C$. Suppose that $a \notin B = \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$ and let $\delta_0 = \min \{\tfrac{1}{2}\delta, |a - u| : u \in B\}$. Take any real number x with $|x - a| < \delta_0$. If $x \in C$, then by (2.8)

$$|g(x) - g(a)| = |f(x) - f(a)| < \eta,$$
$$|h(x) - h(a)| = |f'_E(x) - f'_E(a)| < \eta.$$

If $x \in G$, then $x \in (a_n, b_n)$ for some $n > N$. We have, using (2.9), $b_n - a_n < \delta$ and $|a_n - a| < \delta$. So by (2.8)

$$|A_n(b_n - a_n)^2| \leqslant |f'_E(b_n) - f'_E(a_n)| + 2\left| \frac{f(b_n) - f(a_n)}{b_n - a_n} - f'_E(a_n) \right| < 3\eta,$$

$$|B_n(b_n - a_n)| \leqslant 3\left| \frac{f(b_n) - f(a_n)}{b_n - a_n} - f'_E(a_n) \right| + |f'_E(b_n) - f'_E(a_n)| < 4\eta,$$

$$|g(x) - g(a)| \leqslant |g(a_n) - g(a)| + |g(x) - g(a_n)|$$
$$\leqslant |f(a_n) - f(a)| + |A_n(b_n - a_n)^2| + |B_n(b_n - a_n)| + K(x - a_n)$$
$$< \eta + 3\eta + 4\eta + K\eta$$
$$< (8 + K)\eta$$

where $K = \sup \{|f'_E(x)| : x \in C\}$.

$$|h(x) - h(a_r)| \leqslant |h(a_n) - h(a)| + |h(x) - h(a_n)|$$

$$\leqslant |f'_E(a_n) - f'_E(a)| + 3|A_n(b_n - a_n)^2| + 2|B_n(b_n - a_n)|$$

$$< \eta + 9\eta + 8\eta$$

$$= 18\eta.$$

This gives that g and h are continuous at a. Let $a = a_r$ for some γ $(1 \leqslant \gamma \leqslant N)$. Then clearly g and h are continuous at a on the right. Take

$$\delta_0 = \min \{\tfrac{1}{2}\delta, \ |u - a| : \ u \in B \text{ and } u \neq a\}.$$

Let x be any real number in $(a - \delta, a)$. If $x \in C$, then $|g(x)| - g(a)| < \eta$ and $|h(x) - h(a)| < \eta$. If $x \in G$, then $x \in (a_n, b_n)$ for some $n > N$. As above we can show that

$$|g(x) - g(a)| < (8 + k)\eta \ \text{and} \ |h(x) - h(a)| < 18\eta.$$

Hence g and h are continuous at a on the left. If $a = b_r$ for some γ $(1 \leqslant \gamma \leqslant N)$, then as above we can show that g and h are continuous at a.

Case II. Let $G = \bigcup_{n=1}^{N} (a_n, b_n)$ for some positive integer N. We may suppose that $a < a_1 < b_1 < a_2 < b_2 < \ldots < a_N < b_N < b$. It is easy to see that g and h are continuous.

By Corollary 2.2.1, $g'(x)$ exists for all $x \in [a, b]$ and $g'(x) = h(x)$. Thus g possesses continuous derivative on the whole real line and $g'(x) = h(x) = f'_E(x)$ for all $x \in C$. This completes the proof of the theorem.

3. Results on approximate strong and approximate uniform differentiability

In this section we first show that a function approximately semi-continuous on a measurable set E is measurable; further if the function is finite a.e. on E, then it is approximately continuous a.e. on E.

Definition 3.1 : An extended real-valued function f on the measurable set E is said to be approximately upper [lower] semi-continuous at $a \in E$ if for every $\varepsilon > 0$ there is a measurable set $S_\alpha \subset E$ having a as point of density such that for all $x \in S_\alpha$,

$$f(x) \leqslant f(a) + \varepsilon \ [f(x) \geqslant f(a) - \varepsilon].$$

If f is approximately continuous at a, then it is obvious that f is approximately upper and lower semi-continuous at a; conversely if f is approximately upper and lower semi-continuous at a and $f(a) \neq \pm \infty$, then f is approximately continuous at a.

Lemma 3.1: Let A be any subset of the real line. If A contains almost all of its density points, then A is measurable.

Proof: Let B denote the complement of A. Assume that A is not measurable. Then the sets A and B are not metrically separated ([3,] Ch. V, p. 117). So by Theorems 5.7 and 5.8 ([3], Ch. V, p. 117) there is a set $E \subset B$ such that $m^* E > 0$ and at each point of E the density of A is unity. This contradicts the hypothesis that A contains almost all of its density points. Hence A is measurable.

Theorem 3.1: (Cf. [2], p. 309). Let f be an extended real-valued function on the measurable set E. If f is approximately upper [lower] semi-continuous on E, then f is measurable on E. Further if f is finite a.e. on E, then f is approximately continuous a.e. on E.

Proof: Suppose that f is approximately upper semi-continuous on E. Let γ be any real number and let $A = \{x : x \in E$ and $f(x) \geqslant \gamma\}$. Let α be a point of density of A and let $\alpha \in E$. Assume that $f(\alpha) < \gamma$. Since f is approximately upper semi-continuous at α, there is a measurable set $S_\alpha \subset E$ having α as point of density such that

$$f(x) < \gamma \text{ for all } x \in S_\alpha. \tag{3.1}$$

Let $B = E - S_\alpha$. From (3.1) we see that $A \subset B$. Clearly the density of B at α is zero which contradicts the fact that A has unit density at α. Hence $f(\alpha) \geqslant \gamma$, that is, $\alpha \in A$. Since A has density zero a.e. on E' (complement of E) it follows that A contains almost all of its density points. So by Lemma 3.1, the set A is measurable. This gives that f is measurable on E.

Further, if f is finite a.e. on E, by Theorem 10.6 ([5], p. 132), f is approximately continuous a.e. on E.

If f is approximately lower semi-continuous on E, then $-f$ is approximately upper semi-continuous on E and the result follows.

Definition 3.2: Let f be a real-valued function on the set E. f is said to be approximately strongly differentiable at $\alpha \in E$, if there is a measurable set $S \subset E$ containing α and having α as point of density such that the double limit

$$\lim_{\substack{(x, y) \to (\alpha, \alpha) \\ x, y \in S \\ x \neq y}} \frac{f(x) - f(y)}{|x - y|} \text{ exists finitely.}$$

The above limit whenever exists (finite or infinite) is denoted by $f_{sp}^*(\alpha)$ and is called the approximate strong derivative of f at α.

It is easy to see that if $f_{sp}^*(\alpha)$ exists, then the approximate derivative $f_{sp}'(\alpha)$ exists and $f'(\alpha) = f_{sp}^*(\alpha)$.

Definition 3.3 : Let f be a real-valued function on the interval I and let $f'_{ap}(x)$ exist finitely at each point of I. f is said to be approximately uniformly differentiable at a point $a \in I$ if for every $\varepsilon > 0$ there is a measurable set $S \subset I$ containing a and having a as point of density such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'_{ap}(x) \right| < \varepsilon$$

for all x, y $(x \neq y)$ in S.

For the remaining part of this section we suppose that I is a fixed finite open interval, f is a real-valued function on I and $f'_{ap}(x)$ exists finitely at each point of I. This gives that f is measurable on I (see Theorem 3.1).

For each $a \in I$ we denote by F_a the family of all measurable sets $S \subset I$ containing a and having a as point of density. For any two points x, y $(x \neq y)$ in I we write

$$\psi(x, y) = \left| \frac{f(y) - f(x)}{y - x} - f'_{ap}(x) \right|.$$

For $a \in I$ and $S \in F_a$, let

$$\cup (S, a) = \sup \{ \psi(x, y) : x, y \in S \text{ and } x \neq y \}.$$

Then clearly $\cup (S_1, a) \leqslant \cup (S_2, a)$ if $S_1 \subset S_2$.

Let $u(a) = \inf \{ \cup (S, a) : S \in F_a \}$.

Theorem 3.2 : The function u is approximately upper semi-continuous on I and hence u is measurable on I.

Proof : Let $a \in I$. Choose any $\varepsilon > 0$. There is an element $S_0 \in F_a$ such that $\cup (S_0, a) \leqslant u(a) + \varepsilon$. Denote by S the set of points of S_0 where the density of S_0 is unity. Then S is measurable and each point of S is a point of density of S. Clearly $S \in F_x$ for each $x \in S$. Take any $x \in S$. Since $S > S_0$.

$$u(x) \leqslant \cup (S, x) \leqslant \cup (S, a) \leqslant u(a) + \varepsilon.$$

This gives that u is approximately upper semi-continuous at a and so on I. By Theorem 3.1, u is measurable on I.

Theorem 3.3 : f is approximately uniformly differentiable at $a \in I$ iff $u(a) = 0$.

Proof : First suppose that f is approximately uniformly differentiable at a. Choose any $\varepsilon > 0$. Then there is a member $S \in F_a$ such that $\psi(x, y) < \varepsilon$ for all x, y $(x \neq y)$ in S. This gives that $\cup (S, a) \leq \varepsilon$. Since $u(a) \leqslant \cup (S, a)$ we get $0 \leqslant u(a) \leqslant \varepsilon$. It follows that $u(a) = 0$.

Next, let $u(a) = 0$. Choose any $\varepsilon > 0$. There is a member $S \in F_a$ such that $\cup (S, a) < \varepsilon$. Since $\psi(x, y) \leqslant \cup (S, a)$ for $x, y (x \neq y)$ in S we get

$$\psi(x, y) < \varepsilon \text{ for all } x, y \text{ in } S \text{ with } x \neq y.$$

Hence f is approximately uniformly differentiable at a.

Theorem 3.4: For each $k > 0$, the measure of the set

$$E = \{x : x \in I \text{ and } u(x) \geqslant k\} \text{ is zero.}$$

Proof: Since u is measurable, the set E is measurable. Assume that $mE > 0$. Again, since $f'_{ap}(x)$ is finite at each point of E, by Theorem 10.8 ([5], Ch. VII, p. 237) f is BVG on E. So there is a sequence of sets $E_1, E_2, E_3, \ldots$ such that $E = \cup_{n=1}^{\infty} E_n$ and f is BV on each E_n. Since f is measurable on E, from Theorem 4.2 ([5], Ch. VII, p. 222) it follows that the sets $E_1, E_2, E_3, \ldots$ may be taken measurable. Since $mE > 0$, $mE_r > 0$ for some positive integer r. Again, since f is BV on E_r, there is a function g, BV on I, such that $g = f$ on E_r ([5], Lemma 4.1, Ch. VII, p. 221). So $f'_{E_r}(x)$ exists finitely a.e. on E_r. Let A denote the set of points x of E_r such that $f'_{E_r}(x)$ exists finitely and x is a point of density of E_r. Then A is dense in itself and $mA = mE_r > 0$ and $f'_A(x)$ exists finitely at each point of A. By Theorem 2.3, there is a perfect set $B \subset A$ such that $mB > 0$ and f is uniformly differentiable on B. Choose any ε with $0 < \varepsilon < K$. Then there is a $\delta > 0$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'_A(x) \right| < \varepsilon \tag{3.2}$$

for all $x, y (x \neq y)$ in B whenever $|x - y| < \delta$.

Let C denote the set of points of B where the density of B is unity. Then $mC = mB > 0$. Let $a \in C$ and $S = (a - \delta, a + \delta) \cap C$. Then $S \in F_a$ and $f'_A(x) = f'_{ap}(x)$ for all $x \in C$. We have from (3.2),

$$\left| \frac{f(y) - f(x)}{y - x} - f'_{ap}(x) \right| < \varepsilon$$

for all $x, y (x \neq y)$ in S. This gives that $U(S, a) \leqslant \varepsilon$. So $0 \leqslant u(a) \leqslant \varepsilon$. This contradicts the hypothesis that $u(x) \geqslant k$ for all $x \in E \supset C$. Hence $mE = 0$.

Theorem 3.5: Let f be a real-valued function on the open interval I and let $f'_{ap}(x)$ exist finitely at each point of I. Then f is approximately uniformly differentiable almost everywhere on I.

Proof: For each positive integer n, let

$$E_n = \{x : x \in I \text{ and } u(x) \geqslant 1/n\} \text{ and } E = \cup_{n=1}^{\infty} E_n. \text{ By Theorem 3.4, } mE_n = 0$$

for $n = 1, 2, 3, \ldots$ and so $mE = 0$. If $x \in I - E$, then $u(x) = 0$. So by Theorem 3.3 f is approximately uniformly differentiable at x. This proves the theorem.

Theorem 3.6 : f is approximately strongly differentiable at $a \in I$ iff f is approximately uniformly differentiable at a.

Proof : First suppose that f is approximately strongly differentiable at $a \in I$. There is a member $S_1 \in F_a$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'_{op}(a) \right| < \varepsilon \tag{3.3}$$

for all $x, y \, (x \neq y)$ in S_1.

Denote by S_2 the set of points of S_1 where the density of S_1 is unity. Then S_2 has unit density at each of its points. Let $\omega \in S_2$. Since $f'_{op}(\omega)$ exists finitely, there is a member $S_3 \in F_\omega$ such that

$$\lim_{\substack{z \to \omega \\ z \in S_3}} \frac{f(z) - f(\omega)}{z - \omega} = f'_{op}(\omega).$$

Take $S_4 = S_2 \cap S_3$. Then $S_4 \in F_\omega$. Let $z \in S_4$ and $z \neq \omega$. Then $z, \omega \in S_1$ and so by (3.3)

$$\left| \frac{f(z) - f(\omega)}{z - \omega} - f'_{op}(a) \right| < \varepsilon.$$

Letting $z \to \omega$ over the set S_4 we have

$$|f'_{op}(\omega) - f'_{op}(a)| \leqslant \varepsilon. \tag{3.4}$$

Let $x, y \in S_2$ and $x \neq y$. Then

$$\left| \frac{f(y) - f(x)}{y - x} - f'_{op}(x) \right| \leqslant \left| \frac{f(y) - f(x)}{y - x} - f'_{op}(a) \right| + |f'_{op}(x) - f'_{op}(a)|$$
$$< 2\varepsilon \, [\text{using } (3.3) \text{ and } (3.4)].$$

This gives that $0 \leqslant u(a) \leqslant U(S_2, a) \leqslant 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary we obtain $u(a) = 0$. Hence f is approximately uniformly differentiable at a.

Next, let f be approximately uniformly differentiable at a. Choose any $\varepsilon > 0$. There is a member $S \in F_a$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'_{op}(x) \right| < \varepsilon/3 \tag{3.5}$$

for all $x, y \, (x \neq y)$ in S.

Let $x \in S$ and $x \neq a$. Then by (3.5),

$$| f'_{ap}(x) - f'_{ap}(a) |$$
$$\leqslant \left| \frac{f(x) - f(a)}{x - a} - f'_{ap}(a) \right| + \left| \frac{f(a) - f(x)}{a - x} - f'_{ap}(x) \right| < \frac{2}{3} \varepsilon. \qquad (3.6)$$

Again, for $x, y \;\; (x \neq y)$ in S,

$$\left| \frac{f(y) - f(x)}{y - x} - f'_{ap}(a) \right| \leqslant \left| \frac{f(y) - f(x)}{y - x} - f'_{ap}(x) \right| + | f'_{ap}(x) - f'_{ap}(a) |$$
$$< \tfrac{1}{3} \varepsilon + \tfrac{2}{3} \varepsilon = \varepsilon \text{ [using (3.5) and (3.6)].}$$

This gives that f is approximately strongly differentiable at a.

From Theorems 3.5 and 3.6 we obtain the following:

Theorem 3.7 : Let f be a real-valued function on the finite open interval I and let $f'_{ap}(x)$ exist at each point of I. Then f is approximately strongly differentiable almost everywhere on I.

4. Essentially BV and AC functions

Definition 4.1 : Let f be a real-valued function on the measurable set E. f is said to be essentially BV [AC] on E if given any $\varepsilon > 0$ there is a measurable set $A \subset E$ with $m(E - A) < \varepsilon$ such that f is BV [AC] on A.

Let E be a measurable set with $mE < +\infty$ and let f be essentially AC on E. Then it is easy to show that f is essentially BV on E. In this section we show that the converse is also true. Further we show that a measurable function BVG on a set E with $mE < +\infty$ is essentially AC on E.

Throughout this section we suppose that E is a measurable set with $mE < +\infty$ and f is a real-valued function measurable on E.

Theorem 4.1 : If f is approximately strongly differentiable a.e. on E, then f is essentially AC on E.

Proof : Denote by the set of points of E where f is approximately strongly differentiable. Then $m(E - B) = 0$.

Let $\varepsilon > 0$ be chosen arbitrarily. Take $\eta = \varepsilon/(1 + mE)$. Let $a \in B$. Then there is a measurable set $S_a \subset B$ containing a and having a as point of density such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'_{ap}(a) \right| < 1$$

for all x, y $(x \neq y)$ in S_a. Then

$$|f(y) - f(x)| \leqslant \{1 + |f'_{ap}(a)|\} |y - x|$$

for all x, y in S_a. This gives that f is AC on S_a. Since a is a point of density of S_a, there is a positive number δ_a such that

$$\frac{m[S_a \cap \triangle_a(\delta)]}{m \triangle_a(\delta)} > 1 - \eta \tag{4.1}$$

for all δ with $0 < \delta \leqslant \delta_a$, where $\triangle_a(\delta) = [a - \delta, a + \delta]$.

Let $F = \{\triangle_a(\delta) : 0 < \delta \leqslant \delta_a \text{ and } a \in B\}$. Then F covers the set B in the sense of Vitali. Hence by Vitali's Theorem ([3], Ch. V, Th. 5.1, p. 110), there is a finite number of pairwise disjoint intervals

$$\triangle_{a_1}(h_1), \ \triangle_{a_2}(h_2), \ \ldots, \ \triangle_{a_N}(h_N) \ (a_i \in B)$$

in the family F such that

$$\sum_{i=1}^{N} m[B \cap \triangle_{a_i}(h_i)] > mB - \eta. \tag{4.2}$$

Write $A_i = S_{a_i} \cap \triangle_{a_i}(h_i)$ $(i = 1, 2, \ldots, N)$ and $A = \cup_{i=1}^{N} A_i$.

By (4.1) we have

$$mA_i > (1 - \eta) m \triangle_{a_i}(hi) \geqslant (1 - \eta) m[B \cap \triangle_{a_i}(h_i)]$$

and

$$mA = \sum_{i=1}^{N} mA_i > (1 - \eta) \sum_{i=1}^{N} m[B \cap \triangle_{ai}(h_i)]$$
$$> (1 - \eta)(mB - \eta) \quad \text{[using (4.2)]}$$
$$> mB - \eta(1 + mB) = mB - \varepsilon = mE - \varepsilon.$$

The function f is AC on each of the sets $A_1, A_2, \ldots, A_N$. Since the intervals $\triangle_{a_1}(h_1)$, $\triangle_{a_2}(h_2), \ldots, \triangle_{a_N}(h_N)$ are pairwise disjoints we can show that f is AC on A. Hence f is essentially AC on E.

Theorem 4.2: If f is BVG on E, then f is essentially AC on E; hence essentially BV on E.

Proof: Denote by B the set of points of E where f is approximately strongly differentiable. Since f is BVG and measurable on E, E can be expressed in the form $E = \cup_{n=1}^{\infty} E_n$, where f is BV on each E_n and each E_n is measurable. For each n, $f'_{E_n}(x)$ exists finitely a.e. on E_n. Choose any $\varepsilon > 0$. By Theorem 2.3 there is a perfect set $C_n \subset E_n$ with

$m(E_n - C_n) < \varepsilon/2^n$ such that f is uniformly differentiable on C_n. Denote by A_n the set of points of C_n where the density of C_n is unity. Then each point of A_n is a point of density of A_n and $m(E_n - A_n) < \varepsilon/2^n$. Let $A = \cup_{n=1}^{\infty} A_n$. Then $A \subset E$ and $m(E - A) < \varepsilon$.

Let $a \in A$. Then $a \in A_n$ for some n. Choose any $\eta > 0$. Since f is uniformly differentiable on C_n there is a $\delta > 0$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'_{sp}(x) \right| < \frac{1}{3}\eta \tag{4.3}$$

for all x, y ($x \neq y$) in C_n. Take $S = (a - \delta, a + \delta) \cap C_n$. Then S is measurable, $a \in S$ and S has unit density at a. If $x \in S$ and $x \neq a$, then using (4.3) we get

$$| f'_{E_n}(x) - f'_{E_n}(a) |$$
$$\leqslant \left| \frac{f(x) - f(a)}{x - a} - f'_{E_n}(a) \right| + \left| \frac{f(a) - f(x)}{a - x} - f'_{E_a}(x) \right| < \frac{2}{3}\eta. \tag{4.4}$$

Therefore for any x, y ($x \neq y$) in S,

$$\left| \frac{f(y) - f(x)}{y - x} - f'_{E_n}(a) \right| \leqslant \left| \frac{f(y) - f(x)}{y - x} - f'_{E_n}(x) \right| + | f'_{E_n}(x) - f'_{E_n}(a) |$$
$$< \eta \text{ [using (4.3) and (4.4)]}.$$

This gives that f is approximately strongly differentiable at a and so $a \in B$. Hence $A \subset B$. We have $E - B \subset E - A$. So $m(E - B) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $m(E - B) = 0$. Thus f is approximately strongly differentiable a.e. on E. By Theorem 4.1, f is essentially AC on E.

Corollary 4.2.1 : If f is essentially BV on E, then f is essentially AC on E.

Proof : For each positive integer n, there is a measurable set $B_n \subset E$ with $m(E - B_n) < 1/n$ such that f is BV on B_n. Let $B = \cup_{n=1}^{\infty} B_n$. Then $B \subset E$ and $m(E - B) = 0$. The function f is BVG on B. So by Theorem 4.2, f is essentially AC on B which gives that f is essentially AC on E.

Corollary 4.2.2 : If $f'_{sp}(x)$ exists finitely a.e. on E, then f is essentially AC on E.

Proof : Let B denote the set of points of E where $f'_{sp}(x)$ exists finitely. Then $m(E - B) = 0$. By Theorem 10.8 ([5], Ch. VII, p. 237) f is BVG on B. Now by Theorem 4.2, f is essentially AC on B and so on E.

Theorem 4.3. If $f'_{sp}(x)$ exists finitely a.e. on E, then given any $\varepsilon > 0$ there is a perfect set $A \subset E$ with $m(E - A) < \varepsilon$ such that $f'_s(x)$ exists finitely at each point of A and that f is uniformly differentiable on A.

Proof: By Corollary 4.2.2, f is essentially BV on E. Choose any $\varepsilon > 0$. There is measurable set $B \subset E$ with $m(E - B) < \frac{1}{2}\varepsilon$ such that f is BV on B. Denote by C the set of the points of B where $f'_B(x)$ exists finitely. Then $m(B - C) = 0$. By Theorem 2.3, there is a perfect set $A \subset C$ with $m(C - A) < \frac{1}{2}\varepsilon$ such that f is uniformly differentiable on A. We have $A \subset C \subset B \subset E$. So $E - A = (E - B) \cup (B - C) \cup (C - A)$ and $m(E - A) \leqslant m(E - B) + M(B - C) + m(C - A) < \varepsilon$.

This proves the theorem.

From Theorems 4.3 and 2.4 we obtain the following:

Theorem 4.4: If $f'_{ap}(x)$ exists finitely a.e. on E, then given any $\varepsilon > 0$, there is a perfect set $A \subset E$ with $m(E - A) < \varepsilon$ and a function g defined on the real line having the following properties.

    (i) g possesses continuous derivative on the real line.

    (ii) $g(x) = f(x)$ and $g'(x) = f'_{ap}(x)$ for all $x \in A$.

5. Notations

a. e. $=$ almost everywhere
iff $=$ if and only if

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