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Short Communication

On the approximation of H-function for anisotropic scattering. II B, phase function: $\varpi\,(1 + x\cos\theta)$

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Abstract

The approximate form d for the H-function for isotropic scattering given by Karanjai and Sen has been extended to the case of anisotropic scattering. The phase function considered here is $\varpi\,(1 + x\cos\theta)$. Comparison of calculated results with the exact values given by Chandrasekhar shows good agreement.

Key words: H-function, anisotropic scattering, radiative transfer.

1. Introduction

When the equation of transfer for a semi-infinite plane-parallel atmosphere is solved, we get the emergent intensity which, in most cases, is expressed in terms of a function, called Chandrasekhar's H-function, which itself is a solution of a nonlinear integral equation, *viz.*,

$$\frac{1}{H(\mu, \varpi)} = 1 - \mu \int_0^1 \frac{H(\mu', \varpi)\,\psi(\mu')}{\mu + \mu'}\, d\mu', \tag{1}$$

In some cases[1], the expression for the emergent intensity involves the H-function within and outside the integral sign. In such a case, a good approximation for the H-function is necessary to minimize the labour of computation[2].

Here, in this paper, we study the extension of an approximate form of H-function for isotropic scattering given by Karanjai and Sen[3], *viz.*,

$$H(\mu, \varpi) = 1 + \frac{a\mu + b\mu^2}{A + 2\mu}, \tag{2}$$

$$A^2 = 1 - 2 \int_0^1 \psi(\mu)\, d\mu$$

a, b being constants to be determined,

to the case of anisotropic scattering with the phase function $\varpi(1 + x \cos\theta)$, according to which $H(\mu, \varpi)$ and $H^{(1)}(\mu, \varpi)$ are defined in terms of characteristic functions

$$\psi(\mu) = \tfrac{1}{2}\varpi\,[1 + x(1 - \varpi)\mu^2], \tag{3}$$

and

$$\psi(\mu) = \tfrac{1}{4} x\varpi\,(1 - \mu^2), \tag{4}$$

respectively.

The H-function satisfies the following relations:

$$a_0 = 1 - A \tag{5}$$

$$Aa_2 + \tfrac{1}{2}a_1^2 = \int_0^1 \psi(\mu)\,\mu^2\, d\mu \tag{6}$$

where

$$a_n = \int_0^1 H(\mu, \varpi)\,\psi(\mu)\,\mu^n\, d\mu.$$

Case I : Computation of $H(\mu, \varpi)$:

Substituting for $H(\mu, \varpi)$ and $\psi(\mu)$ from eqns. (2) and (3) into eqns. (5) and (6) we get

$$a = K - K_2 b. \tag{7a}$$

$$T_1 b^2 + T_2 b + T_3 = 0, \tag{7b}$$

where

$$K_1 = \frac{S}{P_2}\ ; \quad K_2 = \frac{P_3}{P_2}$$

$$S = \frac{2(1 - A)}{\varpi} - P_1\ ; \quad P_1 = 1 + x\,(1 - \varpi)\,\frac{1}{3}$$

$$P_2 = I_1 + x(1 - \varpi)\, I_3\ ; \quad P_3 = I_2 + x(1 - \varpi)\cdot I_4$$

$$I_n = \int_0^1 \frac{\mu^n}{A + 2\mu} d\mu \; ; \quad T_1 = \frac{\tilde{\omega}}{4} (Q_2 - P_3 K_2)^2$$

$$T_2 = A (Q_3 - Q_2 K_2) + \frac{\tilde{\omega}}{2} (R_1 + P_3 K_1)(Q_2 - P_3 K_2)$$

$$T_3 = A (Q_1 + Q_2 K_2) + \frac{\tilde{\omega}}{4} (R_1 + P_3 K_1)^2 - Q_1$$

$$Q_1 = \tfrac{1}{3} + x(1 - \tilde{\omega}) \cdot \tfrac{1}{5} \; ; \quad Q_2 = I_3 + x(1 - \tilde{\omega}) \cdot I_5$$

$$Q_3 = I_1 + x(1 - \tilde{\omega}) I_6 \; ; \quad R_1 = \tfrac{1}{2} + x(1 - \tilde{\omega}) \cdot \tfrac{1}{4}.$$

From eqns. (7*a*) and (7*b*) we get the values of a and b.

2. Second approximation formula

Substituting for $H(\mu, \tilde{\omega})$ from eqn. (2) and for $\psi(\mu)$ from eqn. (3) into eqn. (1), we obtain the second approximation formula as

$$\frac{1}{H(\mu, \tilde{\omega})} = 1 - \frac{\mu\tilde{\omega}}{2} \Big[\{I'_0 + x(1 - \tilde{\omega}) I'_2\} + \frac{a}{2} \{I''_1 + x(1 - \tilde{\omega}) I''_3\}$$
$$+ \frac{b}{2} \{I''_2 + x(1 - \tilde{\omega}) I''_4\} \Big], \tag{8}$$

where,

$$I'_n = \int_0^1 \frac{\mu'^n}{\mu + \mu'} d\mu'$$

$$I''_n = \int_0^1 \frac{\mu'^n}{(\mu + \mu')(A/2 + \mu')} d\mu'$$

Case II : Computation of $H^{(1)}(\mu, \tilde{\omega})$.

Substituting for $H(\mu, \tilde{\omega})$ and $\psi(\mu)$ from eqns. (2) and (4) into eqns. (5) and (6) we get

$$a = K_1 - K_2 b, \tag{9}$$

$$T_1 b^2 + T_2 b + T_3 = 0, \tag{10}$$

where,

$$K_1 = \frac{S}{P_2} \; ; \quad K_2 = \frac{P_3}{P_1}$$

$$S = \frac{4(1-A)}{x\widehat{\omega}} - \frac{2}{3}\,; \quad P_2 = I_1 - I_3\,; \quad P_3 = I_2 - I_4$$

$$I_n = \int_0^1 \frac{\mu^n}{A+2\mu}\,d\mu\,; \quad T_1 = \frac{x\widehat{\omega}}{8}\,(Q_2 - P_3 K_2)^2$$

$$T_2 = A\,(Q_3 - Q_2 K_2) + \frac{x\widehat{\omega}}{4}\left(\frac{1}{4} + P_3 K_1\right)(Q_2 - P_3 K_2)$$

$$T_3 = A\left(\frac{2}{15} + Q_2 K_1\right) + \frac{x\widehat{\omega}}{8}\left(\frac{1}{4} + P_3 K_1\right)^2$$

$$Q_2 = I_3 - I_5\,; \quad Q_3 = I_4 - I_6\,.$$

Equations (9) and (10) give the values of a and b.

3. Second approximation formula

Substituting for $H(\mu, \widehat{\omega})$ from eqns. (2) and for $\psi(\mu)$ from eqn. (4) into eqn. (1) we obtain the second approximation formula as

$$\frac{1}{H^{(1)}(\mu, \widehat{\omega})} = 1 - \frac{\mu x \widehat{\omega}}{4}\left[(I'_0 - I'_2) + \frac{a}{2}(I''_1 - I''_3) + \frac{b}{2}(I''_2 - I''_4)\right], \qquad (11)$$

I'_n s I''_n are as in case I.

The values of H-functions have been calculated with the eqns. (8) and (11) and the results are given in Tables I and II respectively.

Table I

$\mu/\widehat{\omega}$	0·2	0·4	0·6	0·8	0·975
0·0	1·000000	1·000000	1·000000	1·000000	1·000000
0·2	1·046305	1·100224	1·165587	1·251914	1·384126
0·4	1·067829	1·150380	1·256167	1·407106	1·671748
0·6	1·081856	1·184355	1·320669	1·526007	1·924982
0·8	1·091914	1·209372	1·369889	1·621868	2·153177
1·0	1·099533	1·228699	1·408972	1·701393	2·362199

Table II

$\mu/\varpi x$	0·2	0·4	0·6	0·8	1·0
0·0	1·000000	1·000000	1·000000	1·000000	1·000000
0·2	1·014611	1·030110	1·046611	1·064256	1·083219
0·4	1·017916	1·040893	1·063746	1·088543	1·115615
0·6	1·022676	1·047206	1·073882	1·103067	1·135227
0·8	1·024648	1·051435	1·080715	1·112926	1·148638
1·0	1·026066	1·054488	1·085668	1·120109	1·158463

4. Discussion of the results

Comparing our calculated values with the exact values given by Chandrasekhar[4], we see that :

The values of $H(\mu, \varpi)$ are correct up to five significant place for $\varpi = 0\cdot2$ to $0\cdot6$, up to four significant place for $\varpi = 0\cdot8$ and up to three significant place for $\varpi = 0\cdot975$. Our calculated values of $H^{(1)}(\mu, \varpi)$ are correct up to five significant place for $\varpi x = 0\cdot2$ to $0\cdot6$ and up to at least four significant place for $\varpi x = 0\cdot8$ to $1\cdot0$.

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