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Diffraction by a strip under mixed boundary conditions

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Abstract

A direct technique is presented for solving the problem of diffraction by a strip under mixed boundary conditions. Taking account of correct edge conditions, the problem is reduced to two pairs of coupled integral equations by means of a method due to Jones in the theory of Wiener-Hopf technique. The first approximation to the solution of these coupled integral equations is obtained for large values of the width of the strip. An expression for the quantity representing the sum of the absorption and scattering coefficients of the strip has been obtained by using the approximate solution. Higher order approximations are avoided because of the complications in the presentation of the results.

Key words : Diffraction, Wiener-Hopf technique.

1. Introduction

The Wiener-Hopf technique provides an extremely powerful weapon for attacking two-dimensional diffraction problems. Out of all possible approaches towards the reduction of the physical problem at hand to a problem of Wiener and Hopf, Jones' method is the most suitable one as it is direct and straightforward. A beautiful account of Jone's method can be found in the books of Jones[1] and Noble[2]. As is well known[1] the Wiener-Hopf technique provides an exact solution to the problem of diffraction by half-planes under unmixed boundary conditions, whereas the problem of diffraction by a strip leads to a set of integral equations when Wiener-Hopf technique is applied. The literature of the Wiener-Hopf technique and its application is vast now and references to most of the work in this direction can be found in the books of Jones[1] and Noble[2].

Whilst the theory of the Wiener-Hopf technique has become quite well known and handy at present for solving diffraction problems under unmixed boundary conditions, not much effort was made to tackle mixed boundary value problems in diffraction theory by this technique, until Rawlins[3] published his paper in 1975 where he presented a new technique of solving certain coupled integral equations. In his paper Rawlins solved the problem of diffraction of plane waves by a soft-hard half-plane through a technique which he calls 'Ad-hoc'. The reader is referred to Rawlins' paper for previous references and the origin of such mixed boundary value problems in diffraction theory.

In the present paper, we have demonstrated how Jones' direct method of solving two-dimensional diffraction problems can be applied to the present type of mixed boundary value problems for a strip. We find that the problem of diffraction of a plane wave by a soft-hard strip can readily be reduced to two pairs of coupled integral equations of the kind which is very different, as is expected, from the integral equations of the unmixed strip problem described in Jones' book[1]. However, once a technique of finding the solution of the coupled integral equations occurring in the half-plane problem is discovered (cf. Ref. 3), the coupled integral equations of the strip problem can very well be handled for their approximate solutions when the width 'l' of the strip is large. We have obtained a first approximation to the solutions of these equations for general angle of incidence. Higher order approximations can also be derived by the technique of Jones[1] for finding approximate solution of the unmixed strip problem. The results of these approximations will be published at a later stage.

In Section 2, we demonstrate the technique applied to our strip problem where we find an expression for the far-field by the method of steepest descent, by using the first approximation to the solution of our integral equations. Finally, we derive an expression for the sum of the absorption and scattering coefficients of the mixed strip by using a formula of Jones[1]. The derivation of the integral equations has been shown for any general incident wave, whereas solutions of these integral equations have been obtained under the assumption that the incident wave is a plane wave.

2. Formulation and reduction to integral equations

Assuming that the soft-hard strip occupies the portion $-l < x < 0$ of the plane $y = 0$, the mathematical problem of determining the scattered field $v(x, y)$ is that of solving the partial differential equation

$$(\nabla^2 + k^2)v = 0. \tag{2.1}$$

Here k is the wave number and a time-dependent factor $e^{i\omega t}$ is dropped throughout the paper.

Under the boundary conditions :

$$\left.\begin{aligned} v(x, 0^+) &= -u_0(x, 0) \\ \frac{\partial v(x, 0^-)}{\partial y} &= -\frac{\partial u_0(x, 0)}{\partial y} \end{aligned}\right\} \quad (-l < x < 0) \tag{2.2}$$

$u_0(x, y)$ representing the known incident field and $v(x, y)$ the total field, the continuity conditions :

$$\left.\begin{aligned} v(x, 0^+) &= v(x, 0^-) \\ \frac{\partial v(x, 0^+)}{\partial y} &= \frac{\partial v(x, 0^-)}{\partial y} \end{aligned}\right\} \quad (-\infty < x < -l;\ 0 < x < \infty) \tag{2.3}$$

and also the radiation condition at infinity and the edge conditions (cf. ref. 3) at edges $x = -l$ and $x = 0$ respectively, which are given by

$$v(x, 0) \sim 0(x^{1/4}), \quad \frac{\partial v(x, 0)}{\partial y} \sim 0(x^{-3/4}), \quad \text{as} \quad x \to 0^+.$$

To solve this problem by the Wiener-Hopf technique, we assume $k = k_r - ik_i\ (k_i > 0)$ until up to the end when we put $k_i = 0$, and define the following transforms :

$$\left.\begin{aligned} V(s, y) &= V_-(s, y) + V_1(s, y) + V_+(s, y) \\ &= \int_{-\infty}^{\infty} v(x, y)\, e^{-sx}\, dx \end{aligned}\right\} \tag{2.4}$$

and

$$\left.\begin{aligned} P(s, y) &= P_-(s, y) + P_1(s, y) + P_+(s, y) \\ &= \int_{-\infty}^{\infty} \frac{\partial v(x, y)\, e^{-sx}}{\partial y}\, dx \end{aligned}\right\} \tag{2.5}$$

where

$$V_-(s, y) = \int_{-\infty}^{-l} e^{-sx}\, v(x, y)\, dx, \quad V_1(s, y) = \int_{-l}^{0} v(x, y)\, e^{-sx}\, dx,$$

$$V_+(s, y) = \int_{0}^{\infty} v(x, y)\, e^{-sx}\, dx,$$

with similar definitions for P_-, P_1 and P_+.

We now note that the functions V_1 and P_1 are analytical functions of s, whereas because of the edge conditions[3], $V_- \sim 0(e^{sl}/s^{5/4})$ and $P_- \sim 0(e^{sl}/s^{1/4})$ as $|s| \to \infty$ in the left half plane $\sigma < k_i$, and $V_+ \sim 0(1/s^{5/4})$ and $P_+ \sim 0(1/s^{1/4})$ as $|s| \to \infty$ in the right half plane, $\sigma > -k_i$. We also note that V_1 and P_1 are $0(e^{sl})$ as $|s| \to \infty$ in the right half plane, whereas they are $0(1)$ as $|s| \to \infty$ in the left half plane.

Procedures adopted here, in deriving the integral equations, are similar to that of Jones[1] for the unmixed strip problem, with slight modification, and the reader is constantly referred to Jones' book for details. Writing the solution of the transformed p.d.e. (2.1) as (since $\operatorname{Im} \kappa < 0$)

$$\left.\begin{aligned} V(s, y) &= Ae^{-i\kappa y},\ y > 0 \\ &= B\, e^{i\kappa y},\ \ y < 0 \end{aligned}\right\} \quad (\kappa = \sqrt{(s^2 + k^2)}) \tag{2.6}$$

and taking limits as $y \to 0^{\pm}$, we obtain

$$
\begin{aligned}
A &= V_+(s,0) + V_-(s,0) + V_1(s,0^+), \\
B &= V_+(s,0) + V_-(s,0) + V_1(s,0^-), \\
-i\kappa A &= P_+(s,0) + P_-(s,0) + P_1(s,0^+), \\
i\kappa B &= P_+(s,0) + P_-(s,0) + P_1(s,0^-),
\end{aligned}
\tag{2.7}
$$

after utilizing the continuity conditions (2.3). Eliminating $A + B$ and $A - B$ from equation (2.7), we obtain

$$
\begin{aligned}
&\kappa\,[V_1(s,0^+) - V_1(s,0^-)] \\
&\quad = 2i\,[P_+(s,0) + P_-(s,0)] + i\kappa\,[P_1(s,0^+) + P_1(s,0^-)]
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
&[P_1(s,0^-) - P_1(s,0^+)] \\
&\quad = 2i\kappa\,[V_+(s,0) + V_-(s,0)] + i\kappa\,[V_1(s,0^+) + V_1(s,0^-)].
\end{aligned}
\tag{2.9}
$$

Equations (2.8) and (2.9) will be handled through the Wiener-Hopf technique and we shall make use of the following identities which are the A and B eliminants from (2.7) or which can be obtained from (2.8) and (2.9).

$$
\left.
\begin{aligned}
P_1(s,0^+) &= -i\kappa\,[V_+(s,0) + V_-(s,0) + V_1(s,0^+)] \\
&\qquad - [P_+(s,0) + P_-(s,0)] \\
&\text{and} \\
V_1(s,0^-) &= \frac{1}{i\kappa}\,[P_+(s,0) + P_-(s,0) + P_1(s,0^-)] \\
&\qquad - [V_+(s,0) + V_-(s,0)]
\end{aligned}
\right\}.
\tag{2.10}
$$

We have selected that branch of the square root for which

$$
(-s - ik)^{1/2} = -(s + ik)^{1/2}. \tag{2.11}
$$

As described in Jones' book[1] (pp. 602–4), we now write the equations (2.8) and (2.9) in two different forms each, split the necessary functions by the splitting technique of Jones[1] and use Liouville's theorem to obtain

$$
\left.
\begin{aligned}
P_+(s,0) &= -(s + ik)^{1/2}\,X_+(s) \\
e^{-sl}\,P_-(s,0) &= -(s - ik)^{1/2}\,Y_-(s) \\
V_+(s,0) &= -(s + ik)^{-1/2}\,L_+(s) \\
&\text{and} \\
e^{-sl}\,V_-(s,0) &= -(s - ik)^{-1/2}\,M_-(s)
\end{aligned}
\right\},
\tag{2.12}
$$

where

$$X_+(s) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{(w+ik)^{-1/2}\,[P_-(w,0)+\frac{1}{2}(P_1(w,0^+)+P_1(w,0^-))]}{w-s}\,dw, \quad (\sigma > a)$$

$$Y_-(s) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{(w-ik)^{-\frac{1}{2}}\,e^{-wl}}{w-s}\,[P_+(w,0)+\tfrac{1}{2}(P_1(w,0^+)+P_1(w,0))]\,dw, \quad (\sigma < b)$$

$$L_+(s) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{(w+ik)^{1/2}}{w-s}\,[V_-(w,0)+\tfrac{1}{2}(V_1(w,0^+)+V_1(w,0^-))]\,dw, \quad (\sigma > a)$$

and

$$M_-(s) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{(w-ik)^{1/2}\,e^{-wl}}{w-s}\,[V_+(w,0)+\tfrac{1}{2}(V_1(w,0^+)+V_1(w,0^-))]\,dw, \quad (\sigma < b) \qquad (2.13)$$

Next, changing w to $-w$ in $X_+(s)$ and $L_+(s)$, and s to $-s$ in $Y_-(s)$ and $M_-(s)$, taking $b = -a$, noting the result (2.11) and defining

$$\left.\begin{aligned} I_+(s) &= P_+(s,0) + e^{sl}\,P_-(-s,0), \\ H_+(s) &= P_+(s,0) - e^{sl}\,P_-(-s,0), \\ F_+(s) &= V_+(s,0) + e^{sl}\,V_-(-s,0), \\ G_+(s) &= V_+(s,0) - e^{sl}\,V_-(-s,0). \end{aligned}\right\} \qquad (2.14)$$

We obtain, from (2.12), the following integral equations which are valid for $\sigma > -b$:

$$\left.\begin{aligned} (s+ik)^{-1/2}\,I_+(s) &= Q_1(s) + \frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty} \frac{(w-ik)^{-1/2}\,I_+(w)}{w+s}\,e^{-wl}\,dw, \\ (s+ik)^{-1/2}\,H_+(s) &= Q_2(s) - \frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty} \frac{(w-ik)^{-1/2}\,H_+(w)}{w+s}\,e^{-wl}\,dw, \\ (s+ik)^{+1/2}\,F_+(s) &= P_1(s) - \frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty} \frac{(w-ik)^{1/2}\,F_+(w)}{w+s}\,e^{-wl}\,dw, \\ \text{and}\qquad & \\ (s+ik)^{1/2}\,G_+(s) &= P_2(s) + \frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty} \frac{(w-ik)^{1/2}\,G_+(w)}{w+s}\,e^{-wl}\,dw, \end{aligned}\right\} \qquad (2.15)$$

where

$$
\left.\begin{aligned}
P_1(s) &= -\frac{1}{4\pi}\int_{b-i\infty}^{b+i\infty}\frac{(w-ik)^{1/2}}{w+s}\,[(V_1(w,0^+)+V_1(w,0^-))\,e^{-wl}\\
&\qquad +(V_1(-w,0^+)+V_1(-w,0^-))]\,dw,\\
P_2(s) &= -\frac{1}{4\pi}\int_{b-i\infty}^{b+i\infty}\frac{(w-ik)^{1/2}}{w+s}\,[(V_1(w,0^+)+V_1(w,0^-))\,e^{-wl}\\
&\qquad -(V_1(-w,0^-)+V_1(-w,0^-))]\,dw,\\
Q_1(s) &= -\frac{1}{4\pi}\int_{b-i\infty}^{b+i\infty}\frac{(w-ik)^{-1/2}}{w+s}\,[(P_1(w,0^+)+P_1(w,0^-))\,e^{-wl}\\
&\qquad +(P_1(-w,0^-)+P_1(-w,0^-))]\,dw,\\
Q_2(s) &= -\frac{1}{4\pi}\int_{b-i\infty}^{b+i\infty}\frac{(w-ik)^{-1/2}}{w+s}\,[(P_1(w,0^+)+P_1(w,0^-))\,e^{-wl}\\
&\qquad -(P_1(-w,0^+)+P_1(-w,0^-))]\,dw
\end{aligned}\right\}\qquad (2.16)
$$

We note that the functions P_1, P_2, Q_1 and Q_2 are not completely known for the type of boundary value problem we are handling. However, if we make use of (2.10) in (2.16) and evaluate some of the integrals by closing the contour in appropriate half-planes and combine the results with the left hand sides of the equation (2.15), the equation (2.15) can be expressed as:

$$
\begin{aligned}
(s+ik)^{-1/2}\,I_+(s) &= R_1(s)\\
&\quad +\frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty}\frac{dw}{w+s}\,[I_+(w)\,(w-ik)^{-1/2}\,e^{-wl}-i\,(w+ik)^{1/2}\,F_+(-w)],\\
(s+ik)^{1/2}\,F_+(s) &= S_1(s)\\
&\quad -\frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty}\frac{dw}{w+s}\,[F_+(w)\,(w-ik)^{1/2}\,e^{-wl}-i\,(w+ik)^{-1/2}\,I_+(-w)],\\
(s+ik)^{-1/2}\,H_+(s) &= R_2(s)\\
&\quad -\frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty}\frac{dw}{w+s}\,[H_+(w)\,(w-ik)^{-1/2}\,e^{-wl}+i\,(w+ik)^{1/2}\,G_+(-w)],\\
(s+ik)^{1/2}\,G_+(s) &= S_2(s)\\
&\quad +\frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty}\frac{dw}{w+s}\,[G_+(w)\,(w-ik)^{1/2}\,e^{-wl}+i\,(w+ik)^{-1/2}\,H_+(-w),
\end{aligned}
\qquad (2.17)
$$

where,

$$R_1(s) = \frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty} \frac{dw}{w+s}\,[(w-ik)^{-1/2}(P_1(w,0^-)\,e^{-wl} + P_1(-w,0^-))$$
$$- i(w+ik)^{1/2}(V_1(w,0^+)e^{-wl} + V_1(-w,0^+))],$$

$$R_2(s) = -\frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty} \frac{dw}{w+s}\,[(w-ik)^{-1/2}(P_1(w,0^-)\,e^{-wl} - P_1(-w,0^-))$$
$$- i(w+ik)^{1/2}(V_1(w,0^+)\,e^{-wl} - V_1(-w,0^+))],$$

$$S_1(s) = -\frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty} \frac{dw}{w+s}\,[(w-ik)^{1/2}(V_1(w,0^+)\,e^{-wl} + V_1(-w,0^+))$$
$$- i(w+ik)^{-1/2}(P_1(w,0^-)\,e^{-wl} + P_1(-w,0^-))],$$

and

$$S_2(s) = \frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty} \frac{dw}{w+s}\,[(w-ik)^{1/2}(V_1(w,0^+)\,e^{-wl} - V_1(-w,0^+))$$
$$- i(w+ik)^{-1/2}(P_1(w,0^-)\,e^{-wl} - P_1(-w,0^-))]. \tag{2.18}$$

The functions R_1, R_2, S_1, S_2 in (2.18) are completely known by means of the boundary conditions (2.2) and the equations (2.17) are the desired pairs of coupled integral equations for our strip problem. These equations are best solved by the method of successive approximation. In what follows, we shall obtain the first approximation to the solution of the equations (2.17) in the case of incident plane wave, assuming l to be large.

3. Approximate solution—Incident plane wave

We now assume that the incident wave is a plane pulse, given by :

$$u_0(x,y) = \exp[-ik(x\cos\phi_0 + y\sin\phi_0)], \qquad (0 < \phi_0 < \pi/2). \tag{3.1}$$

We then have

$$V_1(s,0^+) = \frac{1}{s+ik\cos\phi_0}\,[1-\exp(s+ik\cos\phi_0)\,l]$$

and

$$P_1(s,0^-) = -\frac{ik\sin\phi_0}{s+ik\cos\phi_0}\,[1-\exp(s+ik\cos\phi_0)\,l] \tag{3.2}$$

Then choosing $b=c$, where $k_i > c > k_i\ \cos\phi_0$, we can evaluate all the integrals involving the known functions multiplied by $(w+ik)^{1/2}$ or $(w+ik)^{-1/2}$, by closing the contours on the left.

Hence, if we take the new unknowns as:

$$\lambda_+(s) = L_+(s) - ik\sin\phi_0\left(\frac{1}{s+ik\cos\phi_0} + \frac{\exp(ikl\cos\phi_0)}{s-ik\cos\phi_0}\right)$$

$$\mu_+(s) = F_+(s) + \frac{1}{s+ik\cos\phi_0} + \frac{\exp(ikl\cos\phi_0)}{s-ik\cos\phi_0},$$

$$\nu_+(s) = H_+(s) - ik\sin\phi_0\left(\frac{1}{s+ik\cos\phi_0} - \frac{\exp(ikl\cos\phi_0)}{s-ik\cos\phi_0}\right)$$

$$\theta_+(s) = G_+(s) + \frac{1}{s+ik\cos\phi_0} - \frac{\exp(ikl\cos\phi_0)}{s-ik\cos\phi_0}, \tag{3.3}$$

the integral equations (2.17) take the following forms :

$$\left.\begin{aligned}
&(s+ik)^{-1/2}\lambda_+(s) = l_1(s) + \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{w+s}\,[(w-ik)^{-1/2}\\
&\qquad \times \lambda_+(w)\,e^{-wl} - i\,(w+ik)^{1/2}\mu_+(-w)],\\
&(s+ik)^{1/2}\mu_+(s) = l_2(s) - \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{w+s}\,[(w-ik)^{1/2}\mu_+(w)\,e^{-wl}\\
&\qquad - i\,(w+ik)^{-1/2}\lambda_+(-w)],\\
&(s+ik)^{-1/2}\nu_+(s) = m_1(s) - \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{w+s}\,[(w-ik)^{-1/2}\nu_+(w)\,e^{-wl}\\
&\qquad + i\,(w+ik)^{1/2}\theta_+(-w)],\\
&\text{and}\\
&(s+ik)^{1/2}\theta_+(s) = m_2(s) + \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{w+s}\,[(w-ik)^{1/2}\theta_+(w)\,e^{-wl}\\
&\qquad + i\,(w+ik)^{-1/2}\nu_+(-w)],
\end{aligned}\right\} \tag{3.4}$$

where

$$\left.\begin{aligned}
l_1(s) &= \frac{a_1}{s+ik\cos\phi_0} + \frac{b_1}{s-ik\cos\phi_0},\\
l_2(s) &= \frac{a_2}{s+ik\cos\phi_0} + \frac{b_2}{s-ik\cos\phi_0},\\
m_1(s) &= \frac{a_1'}{s+ik\cos\phi_0} + \frac{b_1'}{s-ik\cos\phi_0},\\
m_2(s) &= \frac{a_2'}{s+ik\cos\phi_0} + \frac{b_2'}{s-ik\cos\phi_0},
\end{aligned}\right\} \tag{3.5}$$

with

$$\left.\begin{aligned} a_1 &= -(2ik)^{1/2}\cos\frac{\phi_0}{2} = a_1' \\ b_1 &= -(2ik)^{1/2}\sin\frac{\phi_0}{2}\exp(ikl\cos\phi_0) = -b_1' \\ a_2 &= (2ik)^{1/2}\sin\frac{\phi_0}{2} = a_2' \\ &\text{and} \\ b_2 &= +(2ik)^{1/2}\cos\frac{\phi_0}{2}\exp(ikl\cos\phi_0) = -b_2'. \end{aligned}\right\} \tag{3.6}$$

We note that $m_1 = 0 = m_2$ when $\phi_0 = \pi/2$. The equations (3.4) are in their most convenient forms for finding the approximate solutions for large l. When l is large, the integrals in (3.4) involving the factor $e^{-\kappa l}$ can be expected to be very small (*see* Watson's Lemma[1]) and neglecting such small terms, the first approximation to the solution of the equations (3.4) can be obtained by solving the following integral equations, once again coupled :

$$(s+ik)^{-1/2}\lambda_+(s) = l_1(s) + \frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty}\frac{\mu_+(w)}{w-s}(w-ik)^{1/2}\,dw,$$

$$(s+ik)^{1/2}\mu_+(s) = l_2(s) + \frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty}\frac{\lambda_+(w)}{w-s}(w-ik)^{-1/2}\,dw,$$

$$(s+ik)^{-1/2}\nu_+(s) = m_1(s) + \frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty}\frac{\theta_+(w)}{w-s}(w-ik)^{1/2}\,dw,$$

and

$$(s+ik)^{1/2}\theta_+(s) = m_2(s) + \frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty}\frac{\nu_+(w)}{w-s}(w-ik)^{-1/2}\,dw, \tag{3.7}$$

We now assume that the solutions of (3.7) can be expressed as (*cf.* ref. 3) :

$$\lambda_+(s) = \frac{\lambda_+^{(0)}(\sqrt{s+ik})}{s+ik\cos\phi_0} + \frac{\lambda_+^{(1)}(\sqrt{s+ik})}{s-ik\cos\phi_0},$$

$$\mu_+(s) = \frac{\mu_+^{(0)}(\sqrt{s+ik})}{s+ik\cos\phi_0} + \frac{\mu_+^{(1)}(\sqrt{s+ik})}{s-ik\cos\phi_0},$$

$$\nu_+(s) = \frac{\nu_+^{(0)}(\sqrt{s+ik})}{s+ik\cos\phi_0} + \frac{\nu_+^{(1)}(\sqrt{s+ik})}{s-ik\cos\phi_0},$$

$$\theta_+(s) = \frac{\theta_+^{(0)}(\sqrt{s+ik})}{s+ik\cos\phi_0} + \frac{\theta_+^{(1)}(\sqrt{s+ik})}{s-ik\cos\phi_0}, \tag{3.8}$$

Then, proceeding in a way similar to the one described in Rawlins' paper[3] and remembering that $k_i > c > k_i \cos\phi_0$, the problem of solving the integral equation (3.7) can be reduced to the following problem of determining the functions $\lambda_+^{(0)}$, $\lambda_+^{(1)}$, etc.

To solve

$$i\rho_-(\gamma) = \gamma\sqrt{(\gamma^2 - 2ik)}\chi_-(-\gamma), \tag{3.9}$$

where $\gamma = (s + ik)^{1/2}$ and the pair of functions $\rho_+(\gamma)$ and $\chi_+(\gamma)$ are to be replaced by the pair $(\lambda_+^{(0)}(\gamma), \mu_+^{(0)}(\gamma))$, $(\lambda_+^{(1)}(\gamma), \mu_+^{(1)}(\gamma))$, $(\nu_+^{(0)}(\gamma), \theta_+^{(0)}(\gamma))$ and $(\nu_+^{(1)}(\gamma), \theta_+^{(1)}(\gamma))$ for the purpose of writing the equations for these unknowns. We also have to satisfy the following conditions relating the unknown functions :

$$\lambda_+^{(0)}(\sqrt{2ik}\sin\tfrac{1}{2}\phi_0) = a_1(2ik)^{1/2}\sin\tfrac{1}{2}\phi_0 = -ik\sin\phi_0$$

$$\begin{aligned}\lambda_+^{(1)}(\sqrt{2ik}\cos\tfrac{1}{2}\phi_0) &= b_1(2ik)^{1/2}\cos\tfrac{1}{2}\phi_0\\ &= -ik\sin\phi_0\exp(ikl\cos\phi_0)\end{aligned}$$

$$\mu_+^{(0)}(\sqrt{2ik}\sin\tfrac{1}{2}\phi_0) = \frac{a_2}{(2ik)^{1/2}\sin\tfrac{1}{2}\phi_0} = 1$$

$$\begin{aligned}\mu_+^{(1)}(\sqrt{2ik}\cos\tfrac{1}{2}\phi_0) &= \frac{b_2}{(2ik)^{1/2}\cos\tfrac{1}{2}\phi_0}\\ &= ik\sin\phi_0\exp(ikl\cos\phi_0)\end{aligned}$$

and

$$\nu_+^{(0)}(\sqrt{2ik}\sin\tfrac{1}{2}\phi_0) = a_1'(2ik)^{1/2}\sin\tfrac{1}{2}\phi_0 = -ik\sin\phi_0$$

$$\nu_+^{(1)}(\sqrt{2ik}\cos\tfrac{1}{2}\phi_0) = b_1'(2ik)^{1/2}\cos\tfrac{1}{2}\phi_0 = ik\sin\phi_0\exp(ik\,l\cos\phi_0)$$

$$\theta_+^{(0)}(\sqrt{2ik}\sin\tfrac{1}{2}\phi_0) = \frac{a_2'}{(2ik)^{1/2}\sin\tfrac{1}{2}\phi_0} = 1$$

$$\theta_+^{(1)}(\sqrt{2ik}\cos\tfrac{1}{2}\phi_0) = \frac{b_2'}{(2ik)^{1/2}\cos\tfrac{1}{2}\phi_0} = -ik\sin\phi_0\exp(ikl\cos\phi_0). \tag{3.10}$$

A method of obtaining the solution of (3.9) satisfying (3.10) and the edge-conditions has been described in (4). Leaving aside the details, we obtain the unique solution of the equation (3.9) as

$$\lambda_+^{(0)}(\gamma) = [A_{-1}^{(0)} + A_0^{(0)}\gamma]\sqrt{(\gamma + \sqrt{2ik})},$$

$$\mu_+^{(0)}(\gamma) = [A_{-1}^{(0)} - A_0^{(0)}\gamma]/[\gamma\sqrt{(\gamma + \sqrt{2ik})}],$$

$$\lambda_+^{(1)}(\gamma) = [A_{-1}^{(1)} + A_0^{(1)}\gamma]\sqrt{(\gamma + \sqrt{2ik})},$$

$$\mu_+^{(1)}(\gamma) = [A_{-1}^{(1)} - A_0^{(1)}\gamma]/[\gamma\sqrt{(\gamma + \sqrt{2ik})}],$$

$$\nu_+^{(0)}(\gamma) = [B_{-1}^{(0)} + B_0^{(0)}\gamma]\sqrt{(\gamma + \sqrt{2ik})},$$

$$\theta_+^{(0)}(\gamma) = [B_{-1}^{(0)} - B_0^{(0)}\gamma]/[\gamma\sqrt{(\gamma + \sqrt{2ik})}],$$

$$\nu_+^{(1)}(\gamma) = [B_{-1}^{(1)} + B_0^{(1)}\gamma]\sqrt{(\gamma + \sqrt{2ik})},$$

and

$$\theta_+^{(1)}(\gamma) = [B_{-1}^{(1)} - B_0^{(1)}\gamma]/[\gamma\sqrt{(\gamma+\sqrt{2ik})}], \tag{3.11}$$

where the constants $A_p^{(i)}$ and $B_p^{(i)}$ $(i = 0, 1;\ p = -1, 0)$, determined by the relations (3.10) are given by

$$\begin{aligned}
A_{-1}^{(0)} &= \frac{1}{2}\left[\frac{a_1\xi_0}{\sqrt{(\xi_0+\sqrt{2ik})}} + a_2\sqrt{(\xi_0+\sqrt{2ik})}\right],\\
A_0^{(0)} &= \frac{1}{2\xi_0}\left[\frac{a_1\xi_0}{\sqrt{(\xi_0+\sqrt{2ik})}} - a_2\sqrt{(\xi_0+\sqrt{2ik})}\right],\\
A_{-1}^{(1)} &= \frac{1}{2}\left[\frac{b_1\eta_0}{\sqrt{(\eta_0+\sqrt{2ik})}} + b_2\sqrt{(\eta_0+\sqrt{2ik})}\right],\\
A_0^{(1)} &= \frac{1}{2\eta_0}\left[\frac{b_1\eta_0}{\sqrt{(\eta_0+\sqrt{2ik})}} - b_2\sqrt{(\eta_0+\sqrt{2ik})}\right],\\
B_{-1}^{(0)} &= A_{-1}^{(0)},\quad B_0^{(0)} = A_0^{(0)},\\
B_{-1}^{(1)} &= -A_{-1}^{(1)},\quad B_0^{(1)} = -A_0^{(1)},
\end{aligned} \tag{3.12}$$

where the constants a_1, a_2, b_1 and b_2 are those given in (3.6) and

$$\xi_0 = (2ik)^{1/2}\sin\tfrac{1}{2}\phi_0$$

and

$$\eta_0 = (2ik)^{1/2}\cos\tfrac{1}{2}\phi_0. \tag{3.13}$$

The transformed field can now be determined by means of the relation (2.6) and the results

$$2A = \mu_+(s) + \theta_+(s) + e^{sl}[\mu_+(-s) - \theta_+(-s)]$$

and

$$2ikB = \lambda_+(s) + \nu_+(s) + e^{sl}[\lambda_+(-s) - \nu_+(-s)] \tag{3.14}$$

which are derived from our earlier substitutions.

We now proceed further to determine the quantities A and B. We have the following results:

$$\begin{aligned}
\mu_+^{(0)}(\gamma) + \theta_+^{(0)}(\gamma) &= 2\,\frac{A_{-1}^{(0)} - A_0^{(0)}\gamma}{\gamma\sqrt{(\gamma+\sqrt{2ik})}},\\
\mu_+^{(1)}(\gamma) + \theta_+^{(0)}(\gamma) &= 0,\\
\lambda_+^{(0)}(\gamma) + \nu_+^{(0)}(\gamma) &= 2\,[A_{-1}^{(0)} + A_0^{(0)'}\gamma]\sqrt{(\gamma+\sqrt{2ik})},\\
\lambda_+^{(1)}(\gamma) + \nu_+^{(2)}(\gamma) &= 0,\\
\theta_+^{(0)}(\gamma') - \mu_+^{(0)}(\gamma') &= 0,
\end{aligned}$$

$$\theta^{(1)}_{+}(\gamma') - \mu^{(1)}_{+}(\gamma') = -2\,[A^{(1)}_{-1} - A^{(1)}_{0}\,\gamma']/(\gamma'\,\sqrt{(\gamma' + \sqrt{2ik})})$$
$$\nu^{(0)}_{+}(\gamma') - \lambda^{(0)}_{+}(\gamma') = 0,$$
$$\nu^{(1)}_{+}(\gamma') - \lambda^{(1)}_{+}(\gamma') = -2\,[A^{(1)}_{-1} + A^{(1)}_{0}\,\gamma')\,\sqrt{(\gamma' + \sqrt{2ik})}, \tag{3.15}$$

where $\gamma' = (-s + ik)^{1/2} = i\,(s - ik)^{1/2}$. Then, the unknowns A and B are finally obtained in the following form :

$$A = \frac{1}{(s + ik\cos\phi_0)}\left[\frac{A^{(0)}_{-1} - A^{(0)}_{0}\,\gamma}{\gamma\,\sqrt{(\gamma + \sqrt{2ik})}} - \frac{(A^{(1)}_{-1} - A^{(1)}_{0}\,\gamma')\,e^{sl}}{\gamma'\,\sqrt{(\gamma' + \sqrt{2ik})}}\right] \tag{3.16}$$

and

$$i\kappa B = \frac{1}{(s + ik\cos\phi_0)}\,[(A^{(0)}_{-1} + A^{(0)}_{0}\,\gamma)\,\sqrt{(\gamma + \sqrt{2ik})}$$
$$- e^{sl}\,(A^{(1)}_{-1} + A^{(1)}_{0}\,\gamma')\,\sqrt{(\gamma' + \sqrt{2ik})}]. \tag{3.17}$$

These expressions can be cast into the forms involving the constants a_1, a_2, b_1 and b_2 by using (3.12) and (3.13). However, we do not give these forms here, and in the next section, we determine the symptotic expression of the far field. Finally, the sum of the absorption and scattering coefficients has been determined by using a formula due to Jones[1].

4. The far-field and the scattering coefficients

Using (3.16), (3.17), (2.6) and the Mellin's inversion formula for the bilateral Laplace transform we determine the following expressions for the diffracted far-field, for large kr, after writing $x = r\cos\phi$ and $|y| = r\sin\phi$, $(0 < \phi < \pi)$:

$$v_d(x, y) \sim A\,(-ik\cos\phi)\,.\left(\frac{k}{2\pi r}\sin^2\phi\right)^{1/2} e^{-ikr + (1/4)\pi i},\ (y > 0) \tag{4.1}$$

and

$$v_d(x, y) \sim M\,(-ik\cos\phi)\left(\frac{k}{2\pi r}\sin^2\phi\right)^{1/2} e^{-(ikr) + (1/4)\pi i},(y < 0) \tag{4.2}$$

where

$$A\,(-ik\cos\phi) = \frac{1}{ik\,(\cos\phi_0 - \cos\phi)}\left[\frac{(A^{(0)}_{-1} - A^{(0)}_{0}\,\xi)}{\xi\,\sqrt{(\xi + \sqrt{2ik})}}\right.$$
$$\left. - \frac{(A^{(1)}_{-1} - A^{(1)}_{0}\,\eta)}{\eta\,\sqrt{(\eta + \sqrt{2ik})}}\exp(-ikl\cos\phi)\right] \tag{4.3}$$

and

$$M\,(-ik\cos\phi) = -\frac{(1/\sin\phi)}{k^2\,(\cos\phi_0 - \cos\phi)}\,[(A^{(0)}_{-1} + A^{(0)}_{0}\,\xi)\,\sqrt{(\xi + \sqrt{2ik})}$$
$$- (A^{(1)}_{-1} + A^{(1)}_{0}\,\eta)\,\sqrt{(\eta + \sqrt{2ik})}\exp(-ikl\cos\phi) \tag{4.4}$$

where

$$\xi = \sqrt{2ik}\sin\phi/2, \quad \eta = \sqrt{2ik}\cos\phi/2.$$

We note that, by using (3.12) and (3.13), we have

$$A_{-1}^{(0)} = \tfrac{1}{2}\left[\xi_0\sqrt{(\xi_0 + \sqrt{2ik})} - \frac{\eta_0\xi_0}{\sqrt{(\xi_0 + \sqrt{2ik})}}\right],$$

$$A_0^{(0)} = -\tfrac{1}{2}\left[\sqrt{(\xi_0 + \sqrt{2ik})} + \frac{\eta_0\xi_0}{\sqrt{(\xi_0 + \sqrt{2ik})}}\right],$$

$$A_{-1}^{(1)} = \tfrac{1}{2}\left[\eta_0\sqrt{(\eta_0 + \sqrt{2ik})} - \frac{\xi_0\eta_0}{\sqrt{(\xi_0 + \sqrt{2ik})}}\right] . \exp(ik\,l\cos\phi_0)$$

$$A_0^{(1)}\eta_0 = -\tfrac{1}{2}[\eta_0\sqrt{(\eta_0 + \sqrt{2ik})} + \xi_0\eta_0/\sqrt{(\eta_0 + \sqrt{2ik})}]\exp(ik\,l\cos\phi_0). \quad (4.5)$$

Using (4.5), the expressions (4.3) and (4.4) can be written down completely.

We shall now obtain the scattering coefficient of the strip under the mixed conditions, considered here, by using Jones' formula (6) (pp. 454-5)[1]:

$$\sigma_S + \sigma_A = -\frac{1}{kl}\operatorname{Im}\left[\int_C \left(u_0\frac{\partial v_d^*}{\partial n} + v_d\frac{\partial u_0^*}{\partial n}\right) ds\right], \quad (4.6)$$

where $u_0(x, y)$ is the incidental field, $v(x, y)$ is the scattered field and stars denote complex conjugates. In (4.6), σ_S is the scattering coefficient, whereas σ_A is the absorption coefficient of the strip, and C is a large circle which completely encloses the strip.

Following Jones' technique involving the method of stationary phase and using the two expressions (4.1) and (4.2) on the top and bottom halves of the circle C, respectively we obtain:

$$\sigma_S + \sigma_A = -\frac{2\sin\phi_0}{l}\operatorname{Re}[A(-ik\cos\phi_0) + M(-ik\cos\phi_0)], \quad (4.7)$$

where by $A(-ik\cos\phi_0)$ and $M(-ik\cos\phi_0)$, we mean the limiting values of the expressions (4.3) and (4.4) as ϕ tends to ϕ_0.

After using (4.5) and after some manipulations, we obtain the following expression for the 'sum of the absorption and scattering coefficients of the mixed strip'

$$\sigma_S + \sigma_A = 4\sin\phi_0 \quad (4.8)$$

For the purpose of comparison, we quote the corresponding expression for the soft strip, derived in Jones' book[1]:

$$\sigma_S = 4\sin\phi_0, \quad (4.9)$$

obtained by using only the first approximation to the solution of Jones' integral equations for the strip-problem.

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