J. Indian Inst. Sci. 62 (A), May 1980, Pp. 101–110

OL and *TOL* array languages

KAMALA KRITHIVASAN AND NALINAKSHI NIRMAL*
Computer Centre, Indian Institute of Technology, Madras 600 036.

Received on May 6, 1980.

Abstract

Two dimensional developmental systems called *OL* array systems (OLAS) and tabled OL array systems (TOLAS) are proposed. These systems reflect the simultaneous growth of every cell in a rectangular array. It is shown that the families of array languages generated by these models are not closed under many of the operations on arrays. These families of developmental array languages are compared with the array languages already known. Growth function of a DOLAS is studied.

Key words: Array languages, developmental array languages, rectangular arrays, AFM operations, pictorial transformations.

1. Introduction

L-systems were introduced by Lindenmayer originally in connection with some problems in theoretical biology[1]. Several studies have been made to extend the development type of generation to two dimensions[2–4] and this paper is another attempt in this effort where the rewriting is simultaneous and new cells are added in the interior of the array. Hence we define *OL* and *TOL* array systems in which parallel rewriting of every symbol in a rectangular array is considered and each symbol is replaced by an array of the same size to avoid distortion of rectangular arrays. This paper is motivated more from the language theory points of view than from the biological point of view.

In section 2, we review some definitions needed for this paper and then we define *OL* and *TOL* array systems and the languages generated by them. In section 3, we give some examples and study some simple properties. In section 4, we discuss the hierarchy among these families and compare them with the array languages already known. In section 5, we investigate the closure properties of the families of array languages under the *AFM* operations and pictorial transformations.

* Department of Mathematics, Madras Christian College, Madras 600 059.

2. Definitions

In this section, we review some definitions needed for this paper. For the definitions of the array languages and matrix languages, the reader is referred to Siromoney *et al*[5,8].

Notation : Let I be an alphabet—a finite nonempty set of symbols. A matrix M_{mn} (or array) over I is an $m \times n$ rectangular array of symbols from I $(m, n \geqslant 1)$ and the dimensions of the matrix M_{mn} is denoted by $| M_{mn} | = (m, n)$. The set of all matrices over I (including λ) is denoted by I^{**} and $I^{++} = I^{**} - \{\lambda\}$.

Definition 2.1 : An OL array system $(OLAS)$ is a 3-tuple $G = (\Sigma, P, \omega)$, where

1. Σ is a finite nonempty set (the alphabet, say, $\Sigma = \{a_1, \ldots, a_k\}$);
2. $\omega \in \Sigma$ is the axiom;
3. P is a finite nonempty subset of $\Sigma \times \Sigma^{**}$ (called the set of productions) such that

$$(\forall_{a_i})_\Sigma \, (\exists \alpha_{i_j})_{\Sigma^{**}} \, (\langle a_i, \alpha_{ij} \rangle \in P).$$

Also $\alpha_{i_j} \in \Sigma^{**}$ is such that $| \alpha_{i_j} | = (m_j, n_j)$ for each $i = 1, 2, \ldots, k$ and j may be from 1 to r, $r \geqslant 1$, *i.e.*, if P contains rules of the type $a_1 \to \alpha_{1_1}, a_1 \to \alpha_{1_2}, \ldots, a_1 \to \alpha_{1_r}$ then P has all the rules $a_i \to \alpha_{i_1}$, $a_i \to \alpha_{i_2}, \ldots, a_i \to \alpha_{i_r}$, $i = 1, \ldots, k$ with $| \alpha_{i_j} | = (m_j, n_j)$, $i = 1, \ldots, k$, $j = 1, \ldots, r$, $r \geqslant 1$ and (m_j, n_j) is fixed for each $j = 1, \ldots, r$, $r \geqslant 1$. The production $\langle a_i, \alpha_{i_j} \rangle$ is usually written as $a_i \to \alpha_{i_j}$.

Definition 2.2 : A tabled OL array system $(TOLAS)$ is a 3-tuple $G = (\Sigma, \mathcal{P}, \omega)$, where Σ and ω are as defined in definition 2.1 and $\mathcal{P}$ consists of a finite set $\{P_1, \ldots, P_f\}$ for $f \geqslant 1$ and each P_i is a finite subset of $\Sigma \times \Sigma^{**}$ called a table with the following two conditions :

1. $(\forall P)_{\mathcal{P}} \, (\forall a)_\Sigma \, (\exists \alpha)_{\Sigma^{**}} \, (\langle a, \alpha \rangle \in P)$.
2. $(\forall a)_\Sigma \, (\exists \langle a, \alpha \rangle)_P$, α's are of the same dimension.

Definition 2.3 : Let

$$u = \begin{matrix} a_{11} \ldots a_{1n} \\ \cdots\cdots\cdots \\ \\ \cdots\cdots\cdots \\ a_{m1} \ldots, a_{mn} \end{matrix} \quad \text{and } v = \begin{matrix} M_{11} \ldots M_{1n} \\ \cdots\cdots\cdots \\ \\ \cdots\cdots\cdots \\ M_{m1} \ldots M_{mn} \end{matrix}, \quad \text{where } a_{ij} \in \Sigma,$$

$M_{ij} \in \Sigma^{**}$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. We write $u \Rightarrow v$ if $a_{ij} \to M_{ij}$ are in a table P in $\mathcal{P}$ (in P) and all the M_{ij}'s are of the same dimension of a $TOLAS$ $(OLAS)$. $\Rightarrow^*$ is the reflexive, transitive closure of $\Rightarrow$.

Definition 2.4 : Let $G = (\Sigma, \mathscr{P}, \omega)$ be a *TOLAS* (*OLAS* where $\mathscr{P} = P$). The language generated by G is defined as $L(G) = \{M/\omega \underset{G}{\Rightarrow}^* M\}$. A language $L \subseteq \Sigma^{**}$ is called a *TOLAL* (*OLAL*) if and only if, there exists a *TOLAS* (*OLAS*) G such that $L = L(G)$. The family of *TOLAL* (*OLAL*) is denoted by $\mathscr{F}$ *TOLAL* ($\mathscr{F}$ *OLAL*).

Definition 2.5 : A *TOLAS* $G = (\Sigma, \mathscr{P}, \omega)$ is called

1. deterministic, if and only if, for each P in $\mathscr{P}$ and each a in Σ, there exists exactly one rule $a \to \alpha$ in P and the system is denoted by *DTOLAS* and the language generated by it is denoted by *DTOLAL*,

2. propagating if there is no table P in $\mathscr{P}$ such that $P = \{a \to \lambda / a \in \Sigma\}$ and the system is denoted by *PTOLAS* and the language generated by it is denoted by *PTOLAL*.

Remark 2.1 : By the completeness condition and the restriction of the size of the arrays, if in a *OLAS* $G = (\Sigma, P, \omega)$, $a \to \lambda$ is in P then for all $a_i \in \Sigma$, $a_i \to \lambda$ is in P. If in a *TOLAS* $G = (\Sigma, \mathscr{P}, \omega)$, $a \to \lambda$ is a rule in some table P, then P consists of only the rules of the form $a_i \to \lambda$ for all $a_i \in \Sigma$.

$$\text{If } u = \begin{matrix} a_{11} \ldots a_{1n} \\ \cdots\cdots \\ \cdots\cdots \\ a_{m1} \ldots a_{mn} \end{matrix}, \text{ where } a_{ij} \in \Sigma, 1 \leq i \leq m, 1 \leq j \leq n.$$

By applying the rules from the table $P = \{a \to \lambda / a \in \Sigma\}$, we get $u \underset{G}{\Rightarrow} \lambda$, the empty matrix.

3. Examples and elementary properties

In this section, we present some examples of *OLAS*, *TOLAS* and languages that can and cannot be generated by them and also discuss some of their elementary properties.

Example 3.1: Let $G_1 = \left(\{a\}, \left\{a \to \begin{matrix} aa \\ aa \end{matrix}\right\}, a\right)$ be an *OLAS*. Then $L(G_1)$ consists of squares of a's of dimension $2^n, n \geqslant 0$.

Example 3.2 : Let $G_2 = \left(\{a, b\}, \{\{a \to b, b \to b\}, \{a \to a, b \to a\}\}, \begin{matrix} aa \\ aa \end{matrix}\right)$ be a *TOLAS*. Then $L(G_2) = \left\{\begin{matrix} aa \\ aa \end{matrix}, \begin{matrix} bb \\ bb \end{matrix}\right\}$.

We state the three lemmas as the proofs are trivial.

Lemma 3.1 : If G is a *POLAS* (*PTOLAS*) and $x \underset{G}{\Rightarrow} y$, then $|y| \geqslant |x|$.

Lemma 3.2 : A finite matrix language which consists of a single array is an *OLAL*.

Lemma 3.3 : If L is an *OLAL* (*TOLAL*) then $L \cup \{\lambda\}$ is also an *OLAL* (*TOLAL*).

We give an arithmetic characterization of *TOLAS*. Let $G = (\Sigma, \mathcal{P}, \omega)$ be a *TOLAS* where $|\omega| = (m, n)$, $\mathcal{P} = \{P_1, \ldots, P_k\}$ and $P_i = \{a_j \rightarrow a_j/a_j \in \Sigma, \; |a_j| = (r_i, s_i)\}, i = 1, 2, \ldots, k$ then any array of $L(G)$ will be of dimension $mr_1^{q_1} \ldots r_k^{q_k} \times n s_1^{q_1} \ldots s_k^{q_k}$, where $q_1, q_2, \ldots, q_k$ are non-negative integers.

Lemma 3.4 : There are some finite matrix languages which are not *OLAL* or *TOLAL*.

Proof : $L = \left\{ \begin{matrix} aa \\ aa \end{matrix} , \begin{matrix} aaa \\ aaa \\ aaa \end{matrix} \right\}$ is a finite matrix language which is not an *OLAL* or *TOLAL* follows from the arithmetic characterization of *TOLAS*.

Lemma 3.5 : $\left\{ a, \begin{matrix} ab \\ ab \end{matrix} \right\}$ is not an *OLAL* (*TOLAL*).

Proof : If possible let $L = \left\{ a, \begin{matrix} ab \\ ab \end{matrix} \right\}$ be a *TOLAL* generated by $G = (\Sigma, \mathcal{P}, \omega)$. As $\lambda \notin L$, G should be a propagating system. Hence $\omega = a$. To get $\begin{matrix} ab \\ ab \end{matrix}$, we must have a table P containing $a \rightarrow \begin{matrix} ab \\ ab \end{matrix}$. To satisfy the completeness condition we must have a rule $b \rightarrow M$ in P where $M \in \{a, b\}^{++}$ and $|M| = (2, 2)$. So arrays which are not in L will be generated by G. Hence L is not a *TOLAL* (*OLAL*).

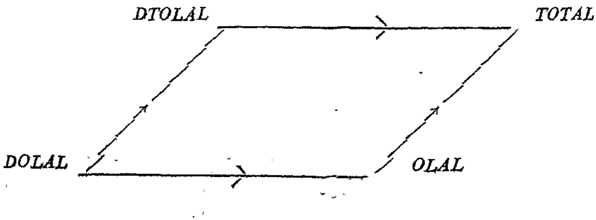
Remark 3.1 : $\{M_{rs}, M_{r_1 s_1}\}$ is not a *TOLAL* (*OLAL*) where $r_1 > r, s_1 \geqslant s$ (or $r_1 \geqslant r$, $s_1 > s$).

Remark 3.2 : $\{M_{r_1 s_1}, M_{r_2 s_2}, \ldots, M_{r_k s_k}\}$ is not a *TOLAL* or an *OLAL*, where at least one r_i, for some i, is such that $r_i > r$, (at least one s_i, for some i, is such that $s_i > s$).

4. Hierarchy and comparison with other array languages

In this section we briefly discuss how the deterministic restriction affects the generative power of *OLAS* and *TOLAS*. We also compare these languages with the array languages already known[4-6].

Theorem 4.1 :



The above-mentioned diagram holds where a solid line denotes strict inclusion (in the direction indicated) and when two families K_1 and K_2 are not connected by a path following the arrows in this diagram, it means that they are incomparable but not disjoint.

Proof: Inclusions follow from definition. For proper inclusion consider a *TOLAL* $L_1 = \left\{\begin{matrix}ab\\ab\end{matrix}, \begin{matrix}bb\\bb\end{matrix}, \begin{matrix}aa\\aa\end{matrix}\right\}$, generated by a *TOLAS* $G_1 = \left(\{a, b\}, \{P_1, P_2\}, \begin{matrix}ab\\ab\end{matrix}\right)$ where $P_1 = \{a \to b, b \to b\}$, $P_2 = \{a \to a, b \to a\}$. If possible let L_1 be generated by an *OLAS* $G' = (\{a, b\}, P, \omega)$. Let $\omega = \begin{matrix}ab\\ab\end{matrix}\left(\omega = \begin{matrix}bb\\bb\end{matrix} \text{ and } \omega = \begin{matrix}aa\\aa\end{matrix} \text{ are analogous}\right)$. If $\omega \Rightarrow \begin{matrix}bb\\bb\end{matrix}$ then we have $\{a \to b, b \to b\} \subset P$. $\begin{matrix}aa\\aa\end{matrix}$ can be derived from ω or from $\begin{matrix}bb\\bb\end{matrix}$. In the former case we have $\{a \to a, b \to a\} \subset P$. Combining these rules with $a \to b$, $b \to b$ we get a derivation $\begin{matrix}ab\\ab\end{matrix} \Rightarrow \begin{matrix}ba\\bb\end{matrix} \notin L_1$. In the latter case we have $b \to a \in P$. Combining this rule with $a \to b$, $b \to b$ we get a derivation $\begin{matrix}ab\\ab\end{matrix} \Rightarrow \begin{matrix}ba\\bb\end{matrix} \notin L_1$. Similarly if $\omega \Rightarrow \begin{matrix}aa\\aa\end{matrix}$ we get words which are not in L_1. Hence L_1 is not an *OLAL*. So $\mathscr{F}OLAL \underset{+}{\subset} \mathscr{F}TOLAL$. From the same example we also conclude that $\mathscr{F}DOLAL \underset{+}{\subset} \mathscr{F}DTOLAL$. L_1 is a *DTOLAL* but not an *OLAL*.

Let

$$L_2 = \left\{\begin{matrix}ab\\ab\end{matrix}, \begin{matrix}bb\\bb\end{matrix}, \begin{matrix}bb\\ab\end{matrix}, \begin{matrix}ab\\bb\end{matrix}\right\}$$

be an *OLAL* generated by an *OLAS* $G_2 = \left(\{a, b\}, \{a \to a, a \to b, b \to b\}, \begin{matrix}ab\\ab\end{matrix}\right)$. It is obvious that L_2 is neither a *DTOLAL* nor a *DOLAL*. Hence the theorem.

By lemma 3.2 and lemma 3.5 we have seen that the family of *OLAL* (*TOLAL*) is incomparable but not disjoint with the family of *FML*. From the arithmetic characterization of the family of *OLAL* and *TOLAL*, we conclude that in any infinite *OLAL*, the length and breadth of the array increase exponentially and not linearly. Whereas in the case of *RML*[5] and $(R : X)\,AL$ $(X = R, CF, CS)$[6], the length or the breadth of the array or both increase linearly.

Hence we have the following theorem.

Theorem 4.2 : (i) $(\mathscr{F}RML - \mathscr{F}FML) \cap \mathscr{F}Y = \phi$; (ii) $(\mathscr{F}(R : X)\,AL - \mathscr{F}RML) \cap \mathscr{F}Y = \phi$, where $X = R$, *CF* or *CS*, $Y = OLAL$ or *TOLAL*.

Theorem 4.3 : $\mathscr{F}OLAL \cap \mathscr{F}(CF : R)\,AL \neq \phi$.

Proof : Squares of X's of side 2^n is an $OLAL$ and also a $(CF : R)$ AL (Siromoney *et al*[8]) generated by an $OLAS$ $G = \left(\{X\}, \left\{X \to \begin{matrix} XX \\ XX \end{matrix}\right\}, X\right)$ and by a $(CF : R)$ AG $G' = (V, I, P, S)$, where $V = \{S\}$, $I = \{X\}$, $P = \{S \to (S \oplus S) \ominus (S \oplus S),\ S \to X\}$ respectively. Thus $\mathscr{F}OLAL$ and $\mathscr{F}(CF : R)$ AL are incomparable but not disjoint.

In extended controlled table L array models[4] growth occurs only along the four edges restricted by a table and controlled by a control set. In $OLAS$ and $TOLAS$ each cell grows and hence these developmental models are incomparable with extended control table L array models.

5. Closure properties

In formal language theory a classical step towards achieving mathematical characterizations of a class of languages is to investigate its closure properties with respect to a number of operations like the AFL operations[7]. In this section we investigate the closure properties of $\mathscr{F}OLAL$ and $\mathscr{F}TOLAL$ under the AFM operations and picture language operations[5]. In one dimension, most of the families of developmental string languages are not closed under any of the AFL operations[7].

We have already given the definitions of row and column catenation for arrays, Now we shall define row star, column star and array homomorphism, H.

Definition 5.1 : A mapping H from I^{++} to $(I')^{++}$ is called a homomorphism if $H(X \oplus Y) = H(X) \oplus H(Y)$ and $H(X \ominus Y) = H(X) \ominus H(Y)$. It is easily seen that a homomorphism is defined only when $H(a) = \{r \times s$ array of terminals from I', a in I, r and a the same for all a in $I\}$. If M is a set of matrices then

$$H(M) = \{H(X)/X \text{ in } M\}.$$

Definition 5.2 : If M is a set of matrices than $\bar{M}$, the complement of $M = I^{**} - M$.

Definition 5.3 : If

$$X = \begin{matrix} a_{11} \ldots, a_{1n} \\ \cdots\cdots\cdots \\ \\ \cdots\cdots\cdots \\ a_{m1} \ldots a_{mn} \end{matrix}$$

then the transpose of X is

$$X^T = \begin{matrix} a_{11} \ldots a_{m1} \\ \cdots\cdots\cdots \\ \\ \cdots\cdots\cdots \\ a_{1n} \ldots a_{mn} \end{matrix}$$

quarter turn of X is

$$Q(X) = \begin{array}{c} a_{m1} \ \ldots \ a_{11} \\ \ldots\ldots\ldots\ldots \\ \\ \ldots\ldots\ldots\ldots \\ a_{mn} \ \ldots \ a_{1n} \end{array}$$

the reflection about the right most vertical is

$$\tilde{X} = \begin{array}{c} a_{1n} \ \ldots, a_{11} \\ \ldots\ldots\ldots\ldots \\ \\ \ldots\ldots\ldots\ldots \\ a_{mn} \ \ldots \ a_{m1} \end{array}$$

the reflection about the base is

$$\underset{\sim}{X} = \begin{array}{c} a_{m1} \ \ldots \ a_{mn} \\ \ldots\ldots\ldots\ldots \\ \\ \ldots\ldots\ldots\ldots \\ a_{11} \ \ldots \ a_{1n} \end{array}$$

and a half-turn is

$$\underset{\sim}{\tilde{X}} = \begin{array}{c} a_{mn} \ \ldots \ a_{m1} \\ \ldots\ldots\ldots\ldots \\ \\ \ldots\ldots\ldots\ldots \\ a_{1n} \ \ldots \ a_{11} \end{array}$$

If M is a set of matrices from I^{++} then

$$M^T = \{X^T/X \text{ in } M\}$$

$$\tilde{M} = \{\tilde{X}/X \text{ in } M\}$$

$$\underset{\sim}{M} = \{\underset{\sim}{X}/X \text{ in } M\}$$

$$\underset{\sim}{\tilde{M}} = \{\underset{\sim}{\tilde{X}}/X \text{ in } M\}.$$

Definition 5.4 : If $X \in \{0, 1\}^{++}$ then X^c (the conjugate of X) is the matrix in which every O in X is replaced by a 1 and every 1 by O.

If M is a set of matrices then $M^c = \{X^c/X \text{ in } M\}$.

Theorem 5.1 : The family of *TOLAL* (*OLAL*) is not closed under union, row catenation, column catenation, row +, column +, array homomorphism *H*, intersection and complementation.

Proof : Since every *OLAL* is a *TOLAL* by definition, in what follows we take an *OLAL* (two *OLALs* if the operation is binary) and show that by the application of the operation under consideration we get a language which is not a *TOLAL*.

(i) *Union* : Let $L_1 = \left\{\begin{matrix} aa \\ aa \end{matrix}\right\}$ and $L_2 = \left\{\begin{matrix} aaa \\ aaa \\ aaa \end{matrix}\right\}$ be two *OLALS*. But by lemma 3.4, it follows that $L_1 \cup L_2$ is not a *TOLAL*.

(ii) *Row cantenation* : Let

$$L_3 = \left\{ a, \begin{matrix} aa \\ aa \end{matrix}, \begin{matrix} aaaa \\ aaaa \\ aaaa \\ aaaa \end{matrix}, \ldots \right\} \text{ and}$$

$L_4 = \{a, aa, aaaa, \ldots\}$ be two *OLALs* generated by

$$G_3 = \left(\{a\}, \left\{a \to \begin{matrix} aa \\ aa \end{matrix}\right\}, a\right) \text{ and } G_4 = (\{a\}, \{a \to aa\}, a)$$

respectively. Then

$$L_3 \ominus L_4 = \left\{ \begin{matrix} a \\ a \end{matrix}, \begin{matrix} aa \\ aa \\ aa \end{matrix}, \begin{matrix} aaaa \\ aaaa \\ aaaa \\ aaaa \\ aaaa \end{matrix}, \ldots \right\}$$

is not a *TOLAL* follows from the arithmetic characterization of *TOLAS*.

(iii) *Column catenation* : Taking L_3 and $T(L_4)$ (The transpose of L_4) as two *OLALs*, we can easily show that $L_3 \oplus T(L_4)$ is not a *TOLAL*.

(iv) *Row* + : Consider $(L_3)_+ = \left\{a, (a)_2, (a)_3, (a)_4, \ldots, \begin{matrix} aa \\ aa \end{matrix}, \begin{pmatrix} aa \\ aa \end{pmatrix}_2, \ldots \right\}$. If possible let there be a *TOLAS* $G' = (\{a\}, \mathcal{P}, \omega)$ such that $L(G') = (L_3)_+$. Then $\omega = a$. To generate words of the type $(a)_p$, p a prime number, we must have a table $\{a \to (a)_p\}$. But the number of primes is infinite. Hence $\mathcal{P}$ should contain an infinite number of tables, which is a contradiction. Hence $(L_3)_+$ is not a *TOLAL*.

(v) *Column* + : Nonclosure under this operation can be similarly proved by considering $(L_3)^+$.

(vi) *Array homomorphism* : Let $L_5 = \left\{ \begin{matrix} ab \\ cd \end{matrix}, \begin{matrix} aabb \\ ccdd \end{matrix}, \ldots \right\}$ be generated by an *OLAS* $G_5 = (\{a, b, c, d\}, \{a \to aa, b \to bb, c \to cc, d \to dd\}, \begin{matrix} ab \\ cd \end{matrix})$. Define an array homomorphism H as :

$$H(a) = \begin{matrix} aa \\ aa \end{matrix}, \; H(b) = \begin{matrix} ab \\ cd \end{matrix}, \; H(c) = \begin{matrix} cc \\ cc \end{matrix}, \; H(d) = \begin{matrix} dd \\ dd \end{matrix}.$$

Hence

$$H(L_5) = \left\{ \begin{matrix} aaab \\ aacd \\ ccdd \\ ccdd \end{matrix}, \begin{matrix} aaaaabab \\ aaaacdcd \\ ccccdddd \\ ccccdddd \end{matrix}, \ldots \right\} = \{M_1, M_2, \ldots\}.$$

If $H(L_5)$ is generated by a *TOLAS* $G' = (\{a, b, c, d\}, \mathscr{P}', \omega')$, then $\omega' = M_1$. If $M_1 \Rightarrow M_2$, then we should have a table which contains rules of the form $a \to aa$, $a \to ab$, $b \to ab$, $c \to cd$, $c \to cc$, $d \to cd$, $d \to dd$, in which case we get arrays which do not belong to $H(L_5)$. Hence $H(L_5)$ is not a *TOLAL*.

(vii) *Intersection* :

$$\text{Let } L_6 = \left\{ a, \begin{matrix} ab \\ ab \end{matrix}, \begin{matrix} abbb \\ abbb \\ abbb \\ abbb \end{matrix}, \ldots \right\} \text{ and } L_7 = \left\{ a, \begin{matrix} ab \\ ab \end{matrix}, \begin{matrix} abab \\ abab \\ abab \\ abab \end{matrix}, \ldots \right\}$$

be two *OLALS*. Then $L_6 \cap L_7 = \left\{ a, \begin{matrix} ab \\ ab \end{matrix} \right\}$ is not a *TOLAL* follows from the remark 3.1.

(viii) *Complementation* : The complement of L_3 is not a *TOLAL* follows from the characterization of *TOLAS*.

Theorem 5.2 : The family of *TOLAL* (*OLAL*) is closed under quarter-turn, transpose, half-turn, reflection about the rightmost vertical, reflection about the base and conjugation.

Proof: Let $G = (\Sigma, P, \omega)$ be an *OLAS*. Consider an *OLAS* $G_1 = (\Sigma, P_1, \omega_1)$ where $\omega_1 = T_1(\omega)$ ($T(A)$ denotes transpose of A). $P_1 = (a \to T(\alpha)/a \to \alpha \text{ in } P)$. Then clearly $L(G_1) = T(L(G))$. The proof for the other operations and for the other family is similar.

In the theory of growth functions only the lengths of the words matter, no attention is paid to the words themselves. We extend this idea to *DOLAS* and find that most of the results of Paz and Salomaa[8] immediately extend to *DOLAS* also. The growth equivalence problem and the problem of growth equivalent axioms will be easily solved in the case of arrays since the production rules are such that the right side is of the same size.

The following theorem follows just as in the case of string languages.

Theorem 5.3 : For any *DOLAS G*, the generating function of its growth function equals $\psi(\omega)$. $(I - Ax)^{-1} . \eta$, where A is the growth matrix.

Proof : Proof is similar to theorem 30 of Paz and Salomaa[8].

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