J. Indian Inst. Sci. 62 (B), June 1980, Pp. 73–81

Solutions of f-gravity coupled to SO(3) gauge field

USHA RAUT
Division of Physics and Mathematical Sciences, Indian Institute of Science, Bangalore 560 012.

Received on April 10, 1980; Revised on June 6, 1980.

Abstract

Using Robertson-Walker coordinates, we examine a system consisting of f-gravity coupled to the SO(3) gauge field. Under certain approximations, we have obtained spherically symmetric and static solutions to the coupled equations. These solutions are found to give interesting results in the asymptotic limits. In particular, we have obtained a Yukawa-like potential for the strong gravity field besides other terms.

Key words: Strong gravity, gauge field, strong interaction

1. Introduction

In a recent paper[1], we had considered a Lagrangian for the strong gravity field coupled to an SO(3) gauge field. Working in a conformally flat space-time, we had obtained spherically symmetric, static solutions for the system. However, owing to our specific choice of the coordinate system, (*i.e.*, one of conformal flatness) we had found that our equations for the f-gravity field and for the SO(3) gauge field had become decoupled. Hence, in the present paper, we have solved the problem using a more general metric, namely, the Robertson-Walker metric, whereby we have found that the equations remain coupled.

At this point we would like to mention very briefly the purpose of attacking the problem in a Robertson-Walker system. In recent years, several authors[2–4] have claimed that the universe and the hadron may follow the same geometrical pattern *i.e.*, hadrons are probably some kind of 'micro-universes'. Now as far as cosmological models are concerned, the Robertson-Walker metric is the most extensively used one, owing to the fact that the form of its line element is based directly on an extension of Mach's original principle, which claims that the inertia of individual bodies is a consequence of all the other bodies in the universe.

Therefore, keeping in view the similarity between the universe and the hadrons, and also the fact that strong gravity is known to govern the internal structure of hadrons[5-8], we have found it worthwhile to investigate strong (f) gravity for the Robertson-Walker system.

We will mention briefly some important characteristics of the Robertson-Walker system.

In keeping with Mach's principle, we need first of all a homogeneous and isotropic three-dimensional sub-space, *i.e.*, the space coordinates must necessarily appear in the line element ds^2 in the spherically symmetric combination.

$$d\sigma^2 = dx'^2 + dx^{2^2} + dx^{3^2}$$

in Cartesian coordinates or equivalently as

$$d\sigma^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2 \theta\, d\phi^2$$

when polar coordinates are employed.

Furthermore, the time-coordinate x° plays the role of a Gaussian time[9] coordinate, which physically means that at any given moment of time, the average velocity of the matter in the micro-universe vanishes in its particular three-space. Thus actually the time-coordinate describes geodesic trajectories which are orthogonal to each corresponding three-space. For this reason, a Robertson-Walker system is also referred to as a co-moving system.

With all these points in mind, we now proceed to calculate the various quantities that enter into the field equations.

2. Lagrangian for the system

In this paper, we consider the metric in its typical form

$$ds^2 = (dx^\circ)^2 - e^{\mu(r)}\, d\sigma^2,$$

where $\mu(r)$ depends only on space coordinates.

We begin by employing an Einstein-type Lagrangian for the spin-2 bosons. The important step is to construct an f-g mixing term which imparts a mass to one of the spin-2 fields, and at the same time maintains general covariance throughout.

Thus we may write the combined action integral for the system as

$$I = I_g + I_f + I_{f-g} + I(fW) \tag{1}$$

where the notations follow closely those used in Ref. 1. For I_g, we have the usual Einstein-Lagrangian,

$$I_g = \int \frac{1}{k_g^2} \sqrt{-g}\, R_g\, d^4x \tag{2}$$

the relevant quantities being constructed from the metric tensor $g_{\mu\nu}$ of the weak (Einsteinian) gravity. Similarly for I_f, we have

$$I_f = \int \frac{1}{k_f^2} \sqrt{-f} R_f \, d^4 x \tag{3}$$

where R_f and $\sqrt{-f}$ are calculated using the strong gravity tensor $f_{\mu\nu}$.

For I_{f-g}, we use the mixing term given by Isham *et al*[10], which results in an emergence of mass of the f-field (in keeping with the finite range nature of the strong interactions)

$$I_{f-g} = - \int \frac{m_f^2}{8k_f^2} \sqrt{-g} \, [f^{\mu\nu} - g^{\mu\nu}] \, [f^{k\lambda} - g^{k\lambda}] \, [g_{k\mu} \, g_{\lambda\nu} - g_{k\lambda} \, g_{\mu\nu}] \, d^4 x. \tag{4}$$

Here m_f is the mass of the f-meson (in units of inverse length).

As done earlier we describe the coupling of this f-g system to the SO(3) gauge field by the appropriate Lagrangian.

$$I(fW) = \int \tfrac{1}{4} \, (-f)^{1/2} f^{\mu\nu} f^{\rho\sigma} F^a_{\mu\rho} F^a_{\nu\sigma} \, d^4 x \tag{5}$$

The $F^a_{\mu\rho}$ denote the field strengths for the SO(3) gauge field and are given by

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + q \epsilon^{abc} W^b_\mu W^c_\nu \tag{6}$$

q being the gauge charge.

Making use of the Wu–Yang ansatz[11], we write

$$W^a_j = \frac{\epsilon^{jal} x^l \, p(r)}{qr^2} \tag{7}$$

ϵ^{jal} being the usual totally antisymmetric tensor of rank 3.

The following constraints have to be imposed for static and spherically symmetric solutions :

$$W_{oa} = 0; \qquad W^a_{ia,\,o} = 0 \tag{8}$$

3. Solutions of the field equations

We begin by neglecting the effect of weak gravity, *i.e.*, the term I_g of eqn. (1). The total action I is then evaluated and the energy functional for the system is calculated using the relation

$$E = - \int I \, d^3 x,$$

After a lengthy calculation, we have finally obtained the energy E as

$$E = \frac{4\pi}{k_f^2} \int \left[-\frac{1}{2} e^{\mu/2} \mu'^2 r^2 + \frac{3m_f^2 r^2}{2} e^{3\mu/2} (1 - 2e^{-\mu} + e^{-2\mu}) \right.$$

$$\left. + \frac{k_f^2}{r^2} \left\{ r^2 e^{-\mu/2} \left(\frac{1}{2r^2} p^2 (p+2)^2 + p'^2 \right) \right\} \right] dr \tag{9}$$

We write down next the usual Euler-Lagrange equations by performing variations with respect to μ and p. The variation

$$\frac{\partial E}{\partial \mu} = \frac{\partial}{\partial r} \left(\frac{\partial E}{\partial \mu'} \right)$$

yields the equation for μ as :

$$\mu'' r^2 + \frac{1}{4} \mu'^2 r^2 + 2\mu' r - \frac{3m_f^2}{2} \left(1 + \frac{1}{2} e^{-\mu} - \frac{3}{2} e^{\mu} \right) r^2$$

$$- \frac{1}{2} \frac{k_f^2}{q^2} e^{-\mu} \left[\frac{1}{2r^2} p^2 (p+2)^2 + p'^2 \right] = 0 \tag{10}$$

Similarly the variation

$$\frac{\partial E}{\partial p} = \frac{\partial}{\partial r} \left(\frac{\partial E}{\partial p'} \right)$$

yields the equation for p as :

$$p'' - \frac{1}{2} p' \mu' = \frac{p(p+1)(p+2)}{r^2} \tag{11}$$

It is easily seen that (10) and (11) are both highly nonlinear differential equations of the second order and as such are difficult to solve. The fact that they are also coupled equations makes the problem of solving them even harder.

Hence in order to simplify the problem, we have made certain approximations. To begin with, we have first solved eqn. (11) in the absence of any coupling, *i.e.*, we have solved the equation

$$p'' = \frac{p(p+1)(p+2)}{r^2} \tag{12}$$

Equation (12) can be recognised as nothing other than the equation for the pure Yang-Mills field in the limit of flat space time. It has the three exact solutions, $p = 0$, $p = -1$ and $p = -2$. Of these, the solutions for $p = 0$ and $p = -2$ can be shown to have finite energies, while the $p = -1$ solution has infinite energy[12].

Now we try and solve eqn. (10) for (and in the vicinity of) these three exact solutions for p.

The eqn. (10) is (on dividing throughout by r^2),

$$\mu'' + \frac{1}{4}\mu'^2 + \frac{2\mu'}{r} - \frac{3m_f^2}{2}\left(1 + \frac{1}{2}e^{-\mu} - \frac{3}{2}e^{\mu}\right)$$
$$- \frac{1}{2}\frac{k_f^2}{q^2}e^{-\mu}\left(\frac{1}{2r^4}p^2(p+2)^2 + \frac{p'^2}{r^2}\right) = 0. \tag{13}$$

Substitute into (13)

$$\psi = re^{\mu/4}$$

then it becomes

$$\frac{4\psi''}{\psi} - \frac{3m_f^2}{2}\left(1 + \frac{1}{2}e^{-\mu} - \frac{3}{2}e^{\mu}\right) - \frac{1}{2}\frac{k_f^2}{q^2}e^{-\mu}$$
$$\left[\frac{1}{2r^4}p^2(p+2)^2 + \frac{p'^2}{r^2}\right] = 0. \tag{14}$$

For the case $p = 0$, we have

$$\psi'' - \frac{3}{8}m_f^2\psi - \frac{3}{8}m_f^2\psi\left[\frac{r^4}{2\psi^4} - \frac{3}{2}\frac{\psi^4}{r^4}\right] = 0. \tag{15}$$

Now consider the equation

$$\psi'' = 0$$

which is got from (15), by neglecting the mass term and the other nonlinear terms. It has a solution of the form,

$$\psi = Ar + B,$$

where A and B are constants of integration.

Upon choosing $B = 0$, $A = 1$, we obtain $\psi = r$ as an asymptotic solution of (15).

In order to make eqn. (15) solvable, we substitute $\psi = r$ in all the nonlinear terms on the right hand side of (15). We then obtain,

$$\psi'' - \frac{3}{8}m_f^2\psi = \frac{3}{8}m_f^2 r\left(\frac{r^4}{2r^4} - \frac{3}{2}\frac{r^4}{r^4}\right)$$

or

$$\psi'' - \frac{3}{8}m_f^2\psi = -\frac{3}{8}m_f^2 r. \tag{16}$$

In order to solve (16) we make use of the appropriate Green's function. Now the Green's function for the equation

$$\psi'' - \frac{3}{8}m_f^2\psi = 0$$

with boundary condition $\psi(0)$ is finite, $\psi(\infty) = 0$ has been worked out as

$$\left.\begin{aligned} G(r, s) &= -\frac{1}{2\sqrt{\frac{3}{8}}\, m_f} \exp\left(-\sqrt{\frac{3}{8}}\, m_f \,|\, r - s \,|\right) \quad 0 \leqslant r < s \\ &= -\frac{1}{2\sqrt{\frac{3}{8}}\, m_f} \exp\left(-\sqrt{\frac{3}{8}}\, m_f \,|\, s - r \,|\right) \quad s < r < \infty \end{aligned}\right\}. \tag{17}$$

The solution for (15) is obtained as

$$\psi = \int G(r, s)\, s\, ds.$$

We perform the integral by splitting it into two parts, according to the definition of the Green's function (17), and finally obtain

$$\psi = r + \frac{a_f \exp\left(-\sqrt{\frac{3}{8}}\, m_f\, r\right)}{2\sqrt{\frac{3}{8}}\, m_f}$$

where $a_f = (k_f M m_f)$ is a constant of integration.

$$\therefore\ e^{\mu/4} = 1 + \frac{a_f \exp\left(-\sqrt{\frac{3}{8}}\, m_f\, r\right)}{2\sqrt{\frac{3}{8}}\, m_f\, r}$$

which gives

$$e^{\mu} \approx 1 + \frac{2a_f \exp\left(-\sqrt{\frac{3}{8}}\, m_f\, r\right)}{\sqrt{\frac{3}{8}}\, m_f\, r}. \tag{18}$$

Equation (18) gives the potential for the strong gravity field for the value $p = 0$. It may be noted that exactly the same solution would result also for $p = -2$.

We now try and solve eqn. (14) for a more general value of the variable p in the region of $p = 0$ (the same thing holds also for values of p in the region of $p = -2$). Consider the eqn. (11), *i.e.*,

$$p'' - \frac{1}{2} p' \mu' = \frac{p(p+1)(p+2)}{r^2}.$$

In the absence of the coupling with the f-gravity field, we have

$$p'' = \frac{p(p+1)(p+2)}{r^2}$$

Upon linearising the above equation, we have

$$p'' = 2p/r^2 \tag{19}$$

The solution of (19) is

$$p = ar^2 + \frac{b}{r} \tag{20}$$

a and b being constants of integration.

$\therefore p' = 2ar$ for large r.

Substitute this value of p' in eqn. (14). Then we have

$$\frac{4\psi''}{\psi} - \frac{3m_f^2}{2}\left(1 + \frac{1}{2}e^{-\mu} - \frac{3}{2}e^{\mu}\right) - \frac{1}{2}\frac{k_f^2}{q^2}e^{-\mu} \times \left[\frac{1}{2r^4}p^2(p+2)^2 + \frac{2a}{r}\right] = 0$$

Near $p = 0$, this becomes, upon simplification,

$$\psi'' - \frac{3m_f^2}{8}\psi\left(1 + \frac{1}{2}e^{-\mu} - \frac{3}{2}e^{\mu}\right) - \frac{\psi}{8}\frac{k_f^2}{q^2}e^{-\mu}\left(\frac{2a}{r}\right) = 0$$

Now we again substitute the asymptotic limit $\psi = r$ for all the nonlinear terms of the equation

$$\psi'' - \frac{3}{8}m_f^2\psi = \frac{3}{8}m_f^2\psi\left(\frac{1}{2}\frac{r^4}{\psi^4} - \frac{3}{2}\frac{\psi^4}{r^4}\right) + \frac{\psi}{8}\frac{k_f^2}{q^2}e^{-\mu}(2a/r) = 0$$

We then obtain

$$\psi'' - \frac{3}{8}m_f^2\psi = -\frac{3}{8}m_f^2 r + \frac{ak_f^2}{4q^2} \tag{21}$$

We solve (21) by making use of the same Green's function given in (17).

Then we have been able to obtain our solution as

$$\psi = r + \frac{a_f \exp\left(-\sqrt{\frac{3}{8}}m_f r\right)}{\sqrt{\frac{3}{8}}m_f} - \frac{2a\,k_f^2}{3q^2 m_f^2}.$$

From the above, we can easily obtain,

$$e^{\mu} \simeq 1 + \frac{2a_f \exp\left(-\sqrt{\frac{3}{8}}m_f r\right)}{\sqrt{\frac{3}{8}}m_f r} - \frac{2a\,k_f^2}{3q^2 m_f^2}(1/r) \tag{22}$$

It is easily seen that (22) is identical to (18) as far as the first two terms go, and the only difference is the additional term

$$-\frac{2a\,k_f^2}{3q^2\,m_f^2}\,(1/r)$$

which arises as a result of the coupling between the SO(3) gauge field and the f-gravity field.

The case $p = -1$ is not considered any further since it corresponds to the case of infinite energy and may not be of any physical significance.

4. Discussion

When there is no coupling of the f-gravity field to the SO(3) gauge field as for the case of $p = 0$ or $p = -2$, we get the potential for the f-gravity field as given by eqn. (18). For very large r, *i.e.*, $r \to \infty$ the second term vanishes, and hence we have $e^{\mu} \to 1$, which is the correct asymptotic limit for flat space-time.

It may be noted that the second term of (18) can be identified with a Yukawa-like potential for the f-gravity field.

Next we have the case for a more general value of p, near $p = 0$. For this case also, we get the two terms in (18), but in addition we have a term given by

$$\frac{-2a\,k_f^2}{3q^2\,m_f^2}\left(\frac{1}{r}\right) \quad \text{(see eqn. 22).}$$

This term is proportional to $1/r$, and arises as a consequence of the SO(3) and f-gravity gauge fields being coupled to each other.

We would like to compare our results with those of Krive and Sitenko[12]. Firstly, they have not obtained the asymptotic limit $e^{\mu} \to 1$ for $r \to \infty$ and secondly, the potential obtained by them does not exhibit an Yukawa-like behaviour as is to be expected for a field equation with mass term.

Further, our solution for the linearised p-equation, namely, $p = ar^2 + b/r$ is more general than theirs, and contains the solutions obtained by them in the asymptotic limits.

A possible extension of the paper is to consider coupling of f-gravity to the SU(3) gauge field.

5. Acknowledgement

The author wishes to express her deep sense of gratitude to Professor K. P. Sinha for numerous stimulating and valuable discussions.

References

1. USHA, C. AND SINHA, K. P. — *Curr. Sci.*, 1979, **48**, 883.
2. SIVARAM, C. AND SINHA, K. P. — *Phys. Lett.*, 1976, 60B, 191.
3. CALDIROLA, P., PAVSIC, M. AND RECAMI, E. — *Lett. Nuovo Cim.*, 1978, **48B**, 205
4. SIVARAM, C. AND SINHA, K. P. — *Phys. Rep.*, 1979, **51**, 111.
5. LORD, E. A., SIVARAM, C. AND SINHA, K. P. — *Lett. Nuovo Cim.* 1974, **11**, 142.
6. SIVARAM, C. AND SINHA, K. P. — *Lett. Nuovo Cim.*, 1973, **8**, 324.
7. LORD, E. A., SINHA, K. P. AND SIVARAM, C. — *Prog. Theor. Phys.*, 1974, **52**, 161.
8. SINHA, K. P. AND SIVARAM, C. — *Proc. of the International Conference on Frontiers of Theoretical Physics*, (Jan. 1977). Ed. F. C. Auluck, L. S. Kothari and V. S. Nanda, MacMillan (India) Ltd., New Delhi, 1978, p. 129.
9. ADLER, R., BAZIN, M. AND SCHIFFER, M. — *Introduction to general relativity*, McGraw-Hill Book Company, 1965, p. 61.
10. ISHAM, C. J., ABDUS SALAM AND STRATHDEE — *J. Phys. Rev.*, **D3**, 1805.
11. WU, T. T. AND YANG, C. N. — *Properties of matter under unusual conditions*, Ed. H. Mark and S. Fernback, Interscience, New York, 1969, p. 349.
12. KRIVE, I. V. AND SITENKO, YU. A. — *J. Phys. A. Maths. and Gen.*, 1977, **10**, 1891.