

Short Communication

An algorithm for ranking the extreme points for a linear fractional objective function

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Abstract

An algorithm for ranking the extreme points of a convex polyhedron in the decreasing order of the value of a linear fractional objective function at these points is given. The algorithm extends the algorithm given by Murty¹ for the case of a linear objective function and utilises the criterion given by Kanti Swarup² for the optimality of a basic feasible solution for a linear fractional programming problem.

Key words: Linear fractional programming, ranking extreme points.

In this note an algorithm for ranking the extreme points of a convex polyhedron in the decreasing order of the value of the objective function at these points is given, the objective function being the ratio of two affine functions. The algorithm extends the algorithm given by Murty¹ for the case of a linear objective function and utilises the criterion given by Kanti Swarup² for the optimality of a basic feasible solution for a linear fractional programming problem. This algorithm will be useful in cases where the extreme point from among a proper subset of all extreme points of the feasible region is to be located, at which the objective function takes its maximum value. Such a situation arises when one wants to find the extreme point with the maximum (minimum) value of the objective function, which is also required to satisfy some additional conditions. An example is the case of extreme point linear fractional programming where one wants to find an extreme point of one convex polyhedron which lies in some other convex polyhedron, for which a linear fractional objective function is maximised.

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Consider the linear fractional programming problem P

$$\text{Maximise } (c^t X + a)/(d^t X + \beta) = f(X)$$

$$\text{subject to } AX = b, X \geq 0$$

where A is an $(m \times n)$ matrix, c, d, X are $(n \times 1)$ column vectors, b is an $(m \times 1)$ column vector and a, β are scalars. Let

$$S = \{X \in R^n \mid AX = b, X \geq 0\}.$$

We shall assume that S is non-empty and bounded and that $d^t X + \beta > 0$ on S .

Here S has a finite number of extreme points and we want to arrange the extreme points in the decreasing order of the value of the objective function at these extreme points.

Let X be any extreme point of S with corresponding basis B . We assume that X is nondegenerate so that there exists only one basis representing X . The adjacent extreme points of X are obtained by introducing a non-basic variable into the basis B and removing a basic variable from B . Different choices of non-basic variables introduced into the basis leads to different adjacent extreme points. Let

$$X_B = B^{-1} b, \quad Z^1 = C_B^t X_B + a, \quad Z^2 = d_B^t X_B + \beta$$

$$Z_j^1 = C_B^t B^{-1} a_j, \quad Z_j^2 = d_B^t B^{-1} a_j, \quad \Delta_j = Z^2 (Z_j^1 - c_j) - Z^1 (Z_j^2 - d_j)$$

where a_j is the j -th column of A .

Kanti Swarup² has proved that (1) X is an optimal solution of the problem P if $\Delta_j \geq 0$ for all j and (2) if a non-basic variable x_j for which $\Delta_j \geq 0$ is introduced into the basis, an adjacent extreme point of X with a lesser value of the objective function is attained. Let

$$B(X) = \{Y \mid Y \text{ is an adjacent extreme point of } X \text{ and } f(Y) \leq f(X)\}.$$

The simplex tableau for any element of $B(X)$ can be obtained from the simplex tableau for X .

If X is degenerate there will be many bases corresponding to X . In this case to get all adjacent extreme points of X , introduction of a non-basic variable and removal of a basic variable must be carried out with respect to every basis representing X .

Based on the above considerations an algorithm to rank the extreme points of S in the decreasing order of the value of $f(X)$ at these points is presented below.

Step 1: Find the optimal extreme point X_1 of S by the method given by Kanti Swarup¹. Set $p = 1$,

Step 2: Find $B(X_p)$. The elements of $B(X_p)$ are obtained by introducing into the basis B_p representing X_p , a non-basic variable x_i for which $\Delta_i > 0$, and removing one basic variable by the simplex procedure. If X_p is degenerate, do this for every basis representing X_p .

Step 3: Let

$$C = \bigcup_{i=1}^p B(X_p) - \{X_1, X_2, \dots, X_p\}.$$

If C is empty the algorithm terminates and we have ranked all the extreme points of S . Otherwise go to step 4.

Step 4: Let X_{p+1} be the element of C which gives the maximum value of the objective function $f(X)$. Set $p = p + 1$ and go to step 2.

This algorithm produces the sequence X_1, X_2, \dots which gives the ranking of the extreme points of S in the decreasing order of the value of $f(X)$ at these points. This algorithm will stop in a finite number of steps since the number of extreme points of S is finite.

Validity of the algorithm

Given any extreme point Y of S , there exists a path $Y = Y_0, Y_1, Y_2, \dots, Y_q = X_1$ from Y to the optimal extreme point X_1 such that (1) Y_i is an adjacent extreme point of Y_{i-1} ($1 \leq i \leq q$) and (2) $f(Y_i) > f(Y_{i-1})$ ($1 \leq i \leq q$). Therefore, if Y is the $(p+1)$ -th best extreme point of S , then Y_1 must be one of X_1, X_2, \dots, X_p . Hence in the algorithm, after ranking the first p best extreme points X_1, X_2, \dots, X_p , the next best extreme point X_{p+1} is selected from among the adjacent extreme points of X_1, X_2, \dots, X_p .

In applications of this algorithm it is enough to rank the extreme points of S only up to the stage where we get an extreme point which satisfies the required conditions.

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