

## Exact computation of a matrix symmetrizer using $p$ -adic arithmetic

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### Abstract

A symmetric solution  $X$  of the matrix equation  $XA = A^t X$  is called a symmetrizer of (an arbitrary matrix)  $A$ . A symmetrizer reduces an apparently nonsymmetric matrix eigenvalue problem into a symmetric one. Here we present an implementation of  $p$ -adic arithmetic on the Datta method for computing a symmetrizer exactly.

**Key words :** Matrix symmetrizer,  $p$ -adic arithmetic, Hessenberg matrix.

### 1. Introduction

A symmetrizer<sup>1-3</sup> of an arbitrary square matrix  $A$  is a symmetric solution  $X$  of the matrix equation  $XA = A^t X$ , where  $t$  indicates the transpose. Symmetrizers are useful in the (nonsymmetric) matrix eigenvalue problems and in the stability problems of Control Theory. Computation of eigenvalues is easier for a symmetric matrix than for a nonsymmetric matrix. A symmetrizer reduces an apparently nonsymmetric eigenvalue problem to a symmetric one as given below.

#### 1.1. A symmetrizer for transformation into a symmetric eigenvalue problem

Let  $A$  be a given nonsymmetric matrix. If there exists a positive definite symmetrizer  $X$  for  $A$  then the nonsymmetric eigenvalue problem

$$Ay = \lambda y \quad (1)$$

can be transformed into a symmetric eigenvalue problem

$$Bx = \lambda x \quad (2)$$

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as shown below:

From (1)

$$Cy = \lambda Xy \quad (3)$$

where  $C = XA$ . Now  $X = S^t DS$  where  $S^t = S^{-1}$  and  $D$  is diagonal.

Let  $G = SCS^t$  and  $F = Sy$ . Then

$$GF = SCS^t Sy = SCy = \lambda DF. \quad (4)$$

If  $d_{ii} > 0$  for all  $i$  then we have  $D = D_1^2$  for some  $D_1$ . Thus we have, from (4),

$$GF = \lambda D_1 D_1 F \text{ or } D_1^{-1} GF = \lambda D_1 F \text{ or } D_1^{-1} G D_1^{-1} D_1 F = \lambda D_1 F.$$

Let  $D_1^{-1} G D_1^{-1} = B$  and  $D_1 F = x$ . Then (1) becomes

$$Bx = \lambda x \quad (5)$$

where  $B$  is symmetric and real.

### 1.2. Symmetrizer for computing zeros of a polynomial

A symmetrizer can be used to obtain the roots of a polynomial equation<sup>4,5</sup>. This is achieved by symmetrizing the companion matrix<sup>3</sup> of the polynomial and then computing its eigenvalues by a matrix method.

### 1.3. Existence of a symmetrizer

There always exists a symmetrizer<sup>1</sup> for a nonderogatory matrix  $A$ . An  $n \times n$  matrix is nonderogatory if and only if its characteristic polynomial (which is of degree  $n$ ) is the same as its minimum polynomial<sup>3</sup>.

### 1.4. Transformation into Hessenberg form

An arbitrary nonsymmetric matrix  $A$  can be transformed into a lower Hessenberg form<sup>3,6</sup>  $B$  ( $A$  and  $B$  having identical eigenvalues since the transformation is of similarity type). It is assumed that  $B$  has nonzero codiagonal. This assumption does not reduce the generality.

### 1.5. Algorithm and arithmetic used

Howland and Farrell<sup>6</sup> were the first to propose a numerical procedure (though not stable in general) for computing a symmetrizer of an arbitrary matrix. Here we consider an algorithm due to Datta<sup>2</sup> for computing a symmetrizer of a real matrix (lower Hessenberg).

It is important to obtain an error-free symmetrizer. Often numerical instability of an algorithm used and errors involved for a given matrix are beyond tolerance. Here we employ  $p$ -adic arithmetic<sup>7,8</sup> for error-free computation of a symmetrizer and describe it for convenience. A criterion for the selection of a prime base  $p$  and a number of digits  $r$  to be involved in representing a quantity (in  $p$ -adic form) is described.

We also describe the computation of the global denominator. Its knowledge is necessary for converting  $p$ -adic output to rational output. It is interesting to note that in the Datta method of symmetrization of an integer (Hessenberg) matrix the global denominator undergoes no modification even during  $p$ -adic multiplication/division.

## 2. Steps for exact computation using $p$ -adic arithmetic

Let there be a numerical problem that involves a finite number of arithmetic operations. Since a digital computer works with finite length numbers in  $n$ -ary scale where  $n$  is any positive integer (say 10) called the radix or base, our input data are always rational numbers. We assume the rational input data to be exact.

*Step 1. Choice of prime base  $p$  and number of digits  $r$ .* Choose a prime base  $p$  and a positive integer  $r$  such that the largest magnitude number (that arises while executing a numerical algorithm for solving the said numerical problem) is less than or equal to  $\sqrt{((p^r - 1)/2)}$ . The condition is sufficient.

*Remarks :* (i) This allows a negative number to be represented in complement form uniquely.

(ii) Choice of  $p$  and  $r$  depends on the given numerical problem and the algorithm to solve it.

(iii) Step 1 requires a method to be devised to obtain a small (preferably the smallest) number greater than or equal to the largest magnitude number (which the algorithm produces).

*Step 2.* Store the  $p$ -adic form of (a positive integral multiple of) the LCM of the denominators (positive) of all the input rational numbers in a location called Global Denominator 'GLODEN'. Encode all the input rational numbers (both negative and positive, assuming a negative sign is attached to a numerator only) to  $p$ -adic numbers.

*Remark :* Since the denominator is a positive integer, its  $p$ -adic representation has an exponent zero.

*Step 3.* Execute (using the  $p$ -adic arithmetic) the numerical algorithm whose input data are: (i) the  $p$ -adic numbers and (ii) the GLODEN. Update the GLODEN, if necessary, whenever there occurs a  $p$ -adic multiplication/division.

*Remarks :* (i) In a  $p$ -adic addition/subtraction GLODEN remains unchanged.

(ii) For the modification of the GLODEN a knowledge of the denominators of the operands is essential. Since the operations are performed in  $p$ -adic arithmetic, the denominators are not explicitly known. Therefore, whenever a modification of the GLODEN is required, the operands must be converted to their rational forms using the current GLODEN.

*Step 4.* Decode the  $p$ -adic output to rational form using the GLODEN.

### 2.1. Encoding rational input to $p$ -adic input (Step 2)

Let  $p$  and  $r$  be known from Step 1. Also, let the usual symbols (like  $+$ ,  $-$ ,  $\dots$ ) of real arithmetic be the symbols for  $p$ -adic operations (add, subtract, multiply, divide) as long as operands are  $p$ -adic numbers.

*Substep 1.* Compute the LCM of the denominators of all rational input data.

*Substep 2.* GLODEN  $\leftarrow$   $p$ -adic form of the LCM. [Let the LCM be  $q_0$ . Then its  $p$ -adic form is  $\dots s_1 s_2 \dots s_r$ , where  $s_i =$  positive remainder of  $q_{i-1} \div p$ ,  $i = 1(1)r$ ; ( $0 \leq s_i < p$ ).  $q_i =$  positive quotient of  $q_{i-1} \div p$ ,  $i = 1(1)r - 1$ .]

*Substep 3.* Convert the numerator and the denominator of each rational input into  $p$ -adic form in the same way as in Substep 2. If the numerator has a negative sign then consider, for the numerator, its complement, viz., " $p^r -$  numerator" before conversion. Divide the  $p$ -adic numerator by the  $p$ -adic denominator using  $p$ -adic division (see  $p$ -adic arithmetic Sec. 2.3).

*Note :* The input data are thus (i) the  $p$ -adic form of all rational input data, and (ii) the GLODEN.

### 2.2. Updating global denominator GLODEN (Step 3)

$p$ -adic addition/subtraction ( $+/-$ ) do not alter the Global Denominator GLODEN.  $p$ -adic multiplication/division, however, may alter the GLODEN.

For each integer  $a$ , where  $0 \leq a \leq p^r - 1$ , define

$$v(a) = a \text{ if } 0 \leq a \leq \frac{p^r - 1}{2}, \quad v(a) = a - p^r \text{ if } \frac{p^r - 1}{2} < a \leq p^r - 1. \quad (6)$$

Also, for each mantissa part  $m_a = \dots s_1 s_2 \dots s_r$  of a  $p$ -adic number  $a = m_a E_a$ , define

$$I(m_a) = \sum_{i=1}^r s_i p^i. \quad (7)$$

(i) *GLODEN after a multiplication :* Let  $a = m_a E_a$  and  $\beta = m_\beta E_\beta$  be two  $p$ -adic numbers to be multiplied. Compute

$$\begin{aligned} \bar{a} &= [v(I(m_a) \cdot \text{GLODEN})] \frac{1}{p^{ra}} \\ \bar{\beta} &= [v(I(m_\beta) \cdot \text{GLODEN})] \frac{1}{p^{r\beta}}. \end{aligned} \quad (8)$$

$$\overline{\text{GLODEN}} = I(\text{GLODEN}).$$

Compute

$$\begin{aligned} \alpha^+ &= \text{GCD}(\bar{a}, \overline{\text{GLODEN}}) \\ \beta^+ &= \text{GCD}(\bar{\beta}, \overline{\text{GLODEN}}) \\ G &= \text{GCD}(\overline{\text{GLODEN}}/\alpha^+, \overline{\text{GLODEN}}/\beta^+). \end{aligned} \quad (9)$$

If  $G = 1$  then GLODEN does not alter. Otherwise,  $\text{GLODEN} \leftarrow \text{GLODEN} \times p$ -adic form of  $G$ .

Remarks : (a)  $\overline{\text{GLODEN}}/\alpha^+$  is the denominator of  $\alpha$ .

(b)  $\overline{\text{GLODEN}}/\beta^+$  is the denominator of  $\beta$ .

(ii)  $\text{GLODEN}$  after a division : Let  $\alpha = m_\alpha E e_\alpha$  and  $\beta = m_\beta E e_\beta$  be two  $p$ -adic numbers.  $\alpha$  is to be divided by  $\beta$ . Here

$$\text{GLODEN} \leftarrow p\text{-adic form of LCM}(\overline{\text{GLODEN}}, \overline{\beta/\beta^+}) \quad (10)$$

where  $\overline{\text{GLODEN}}, \overline{\beta}, \beta^+$  are as defined in (i) of Sec. 2.2.

### 2.3. $p$ -adic arithmetic (Steps 2 and 3)

Let usual symbols like  $+$ ,  $-$ ,  $.$ ,  $/$  represent respectively the  $p$ -adic add, subtract, multiply, and divide operations, as long as the operands are  $p$ -adic numbers. Let

$$\alpha = m_\alpha E e_\alpha, \quad \beta = m_\beta E e_\beta, \quad \gamma = m_\gamma E e_\gamma \quad (11)$$

be three  $p$ -adic numbers where  $m_\alpha, m_\beta$  and  $m_\gamma$  are mantissas and  $e_\alpha, e_\beta, e_\gamma$  are exponents. Also, let  $p$  be the prime base and  $r$  be the number of digits retained in a mantissa.

(i)  $p$ -adic addition ( $\gamma = \alpha + \beta$ ) : Let

$$\begin{aligned} m_\alpha &= .s_1 s_2 \dots s_r \\ m_\beta &= .t_1 t_2 \dots t_r \\ e_\alpha &\geq e_\beta. \end{aligned} \quad (12)$$

Shift  $s_i$ 's by  $e_\gamma = e_\alpha - e_\beta$  places to the left. Then carry out  $p$ -adic addition on these mantissas in the same way as in the finite field arithmetic with prime base  $p$ . The  $p$ -adic point will then be shifted before the leftmost digit of the result. Let this result now be called  $m_\gamma$ . Any carry beyond  $r$  digits is neglected.

Example : Let  $\alpha = .30000000 \text{ E}1$ ,  $\beta = .21313131 \text{ E}0$ . Compute  $\gamma = \alpha + \beta$ .

Here  $p = 5$ ,  $r = 8$ ,  $m_\alpha = .30000000$ ,  $m_\beta = .21313131$ ,  $e_\alpha = 1$ ,  $e_\beta = 0$

$$\begin{aligned} \text{shifted } m_\alpha &= 3.0000000 \\ m_\beta &= .2131313 \\ \hline m_\gamma &= .32131313 \\ e_\gamma &= e_\alpha - e_\beta = 1 - 0 = 1. \end{aligned}$$

Hence  $\gamma = \alpha + \beta = .32131313 \text{ E}1$ .

(ii)  $p$ -adic subtraction ( $\gamma = \alpha - \beta$ ): The process of  $p$ -adic subtraction is similar to that of  $p$ -adic addition.

Example : Let  $\alpha = .30000000 \text{ E}1$   $\beta = .33131313 \text{ E}0$ . Compute  $\gamma = \alpha - \beta$ .

Here  $p = 5$ ,  $r = 8$ ,  $m_\alpha = \cdot 30000000$ ,  $e_\alpha = 1$ ,  $m_\beta = \cdot 21313131$ ,  $e_\beta = 0$

$$\text{shifted } m_\alpha = 3\cdot 0000000$$

$$m_\beta = \cdot 3313131$$

$$m_\gamma = \cdot 32131313$$

$$e_\gamma = e_\alpha - e_\beta = 1 - 0 = 1.$$

Hence  $\gamma = \alpha - \beta = \cdot 32131313$  E1.

(iii) *p*-adic multiplication ( $\gamma = \alpha \cdot \beta$ ): Here exponents are added as in the floating point multiplication (real arithmetic). So  $e_\gamma = e_\alpha + e_\beta$ . Mantissas  $m_\alpha$  and  $m_\beta$  are multiplied in almost the same way as in the finite field arithmetic. The *p*-adic point is shifted before the leftmost digit of the result. This gives  $m_\gamma$ .

*Example*: Let  $\alpha = \cdot 32222222$  E0,  $\beta = \cdot 32131313$  E1. Compute  $\gamma = \alpha \cdot \beta$ .

Here  $p = 5$ ,  $r = 8$ ,  $m_\alpha = \cdot 32222222$ ,  $e_\alpha = 0$ ,  $m_\beta = \cdot 32131313$ ,  $e_\beta = 1$ .

$$m_\alpha = \cdot 32222222$$

$$m_\beta = \cdot 32131313$$

$$42222222$$

$$1000000$$

$$322222$$

$$42222$$

$$3222$$

$$422$$

$$32$$

$$4$$

$$m_\gamma = \cdot 43040404$$

$$e_\gamma = e_\alpha + e_\beta = 0 + 1 = 1.$$

Hence  $\gamma = \alpha \cdot \beta = \cdot 43040404$  E1.

*Note*: In *p*-adic multiplication/division we cannot write  $\cdot 43040404$  E1 =  $4\cdot 3040404$  like a floating point number.

(iv) *p*-adic division ( $\gamma = \alpha/\beta$ ): Here exponents are subtracted as in the floating point division (real arithmetic). So  $e_\gamma = e_\alpha - e_\beta$ .  $m_\alpha$  is divided by  $m_\beta$  in almost the same way as in the finite field arithmetic.

If  $m_\beta$  has  $k$  zeros at the leftmost side then  $m_\alpha$  is divided by the *p*-adic number which is ' $m_\beta$  without the  $k$  zeros'.  $k$  zeros are then appended before the leftmost digit of the quotient and after the *p*-adic point. This gives  $m_\gamma$ .

*Examples*: Let  $\alpha = \cdot 23431013$  E0,  $\beta = \cdot 32131313$  E1. Compute  $\alpha/\beta$ .

Here  $p = 5$ ,  $r = 8$ ,  $m_\alpha = \cdot 23431013$ ,  $e_\alpha = 0$ ,  $m_\beta = \cdot 32131313$ ,  $e_\beta = 1$ .

$$\begin{array}{r}
 m_\beta \quad m_\alpha \quad m_\gamma \\
 \cdot 32131313 \cdot 23431013 \quad (\cdot 41232340 \\
 \quad \quad \quad 20131313 \\
 \hline
 \quad \quad \quad 3300244 \\
 \quad \quad \quad 3213131 \\
 \hline
 \quad \quad \quad 141013 \\
 \quad \quad \quad 103131 \\
 \hline
 \quad \quad \quad 43321 \\
 \quad \quad \quad 42444 \\
 \hline
 \quad \quad \quad 1421 \\
 \quad \quad \quad 1031 \\
 \hline
 \quad \quad \quad 444 \\
 \quad \quad \quad 424 \\
 \hline
 \quad \quad \quad 20 \\
 \quad \quad \quad 20 \\
 \hline
 \quad \quad \quad 0 \\
 \quad \quad \quad 0
 \end{array}$$

$$e_\gamma = e_\alpha - e_\beta = 0 - 1 = -1.$$

Hence  $\gamma = a/\beta = \cdot 41232340 E - 1$ .

Let  $\alpha = \cdot 331313 E0$ ,  $\beta = \cdot 030313 E0$ . Compute  $a/\beta$ .

Here  $p = 5$ ,  $r = 6$ ,  $m_\alpha = \cdot 331313$ ,  $e_\alpha = 0$ ,  $m_\beta = \cdot 030313$ ,  $e_\beta = 0$ .

$$\begin{array}{r}
 m_\beta \quad m_\alpha \quad m_\gamma \\
 \cdot 30313 \cdot 33131 \quad (\cdot 11111 \\
 \quad \quad \quad 30313 \\
 \hline
 \quad \quad \quad 3313 \\
 \quad \quad \quad 3031 \\
 \hline
 \quad \quad \quad 331 \\
 \quad \quad \quad 303 \\
 \hline
 \quad \quad \quad 33 \\
 \quad \quad \quad 30 \\
 \hline
 \quad \quad \quad 3 \\
 \quad \quad \quad 3
 \end{array}$$

$$e_\gamma = e_\alpha - e_\beta = 0 - 0 = 0.$$

Hence  $\gamma = a/\beta = \cdot 011111 E0$ .

*Note* : When there are five digits in the divisor, there is no need to consider more than 5 digits of the dividend.

#### 2.4. Decoding $p$ -adic output to rational form (Step 4)

Let  $a = m_a E e_a$  with GLODEN  $g$  be a  $p$ -adic output. Then

$$\text{rational form of } a = \frac{[v(I(m_a \cdot \bar{g}))] (1/p^a)}{\bar{g}/a^+} \quad (13)$$

(also, see Sec. 2.2) where, for each integer  $a$ ,

$$v(a) = a \text{ if } 0 \leq a \leq \frac{p^r - 1}{2}, \quad v(a) = a - p^r \text{ if } (p^r - 1)/2 < a \leq p^r - 1. \quad (14)$$

For  $m_a = \cdot s_1 s_2 \dots s_r$  of  $a$

$$I(m_a) = \sum_{i=1}^r s_i p^i \quad (15)$$

$$\bar{g} = I(g)$$

$$a^+ = GCD(\bar{a}, \bar{g}). \quad (16)$$

*Note* :  $\bar{g}/a^+$  is the denominator of  $a$  (in rational form).

### 3. The Datta algorithm

Let  $B = (b_{ij})$  be the given  $n \times n$  lower Hessenberg matrix with nonzero codiagonal. Let  $x_1, x_2, \dots, x_n$  be the rows of a symmetrizer  $X$  to be computed. Then

*Step 1* : Choose  $x_n \neq 0$  arbitrarily. (To ensure nonsingularity of  $X$ , choose an  $x_n$  of the form  $(c \ 0 \ 0 \ \dots \ 0)$  where  $c \neq 0$ ).

*Step 2* : Compute  $x_{n-1}, x_{n-2}, \dots, x_1$  recursively from

$$x_i = \frac{1}{b_{i, i+1}} (x_{i+1} B - b_{i+1, i+1} x_{i+1} - b_{i+2, i+1} x_{i+2} - \dots - b_{n, i+1} x_n) \quad (17)$$

$$i = n-1, n-2, \dots, 2, 1.$$

*Step 3* : Print  $x_1, x_2, \dots, x_n$  and stop.

### 4. Choice of prime base $p$ and number of digits $r$

Let the lower Hessenberg matrix  $B$  have (negative and/or positive) integer elements with nonzero codiagonal. This does not reduce the generality. From the Datta algorithm (Sec. 3).

$$x_{n-1} = \frac{1}{b_{n-1, n}} (x_n B - b_{nn} x_n).$$

Choose  $x_n = (1 \ 0 \ 0 \ \dots \ 0)$ .  $L_1$ -norm<sup>3</sup> of  $x_{n-1}$ , viz.,

$$\|x_{n-1}\|_1 \leq 2 \left( \max_j \sum_{i=1}^n |b_{ij}| \right) / b_{n-1, n} \text{ since } | \text{every element of } x_n | \leq 1. \quad (18)$$



Thus the largest magnitude element in  $x_{n-1}$  is less than or equal to the right hand side expression in (18). The factor 2 is due to the addition or subtraction of two numbers, viz.,  $x_n B$  and  $b_{nn} x_n$ .

Again from the Datta algorithm

$$x_{n-2} = \frac{1}{b_{n-2, n-1}} (x_{n-1} B - b_{n-1, n-1} x_{n-1} - b_{n, n-1} x_n).$$

Hence,

$$\|x_{n-2}\|_1 \leq 3 \left( \max_j \sum_{i=1}^n |b_{ij}| \right)^2 / (b_{n-1, n} b_{n-2, n-1}). \quad (19)$$

Similarly,

$$\|x_{n-3}\|_1 \leq 4 \left( \max_j \sum_{i=1}^n |b_{ij}| \right)^3 / (b_{n-1, n} b_{n-2, n-1} b_{n-3, n-2}). \quad (20)$$

Hence, choose  $p$  and  $r$  such that

$$\sqrt{(p^r - 1)/2} \geq n \left( \max_j \sum_{i=1}^n |b_{ij}| \right)^{n-1}. \quad (21)$$

*Note:* The choice of  $p$  and  $r$  is not unique. However,  $p$  and  $r$  are so chosen that  $\sqrt{(p^r - 1)/2}$  is the smallest positive real number satisfying (21).

### 5. Invariance of global denominator

In the Datta algorithm a global denominator GLODEN (for an integer matrix) can be computed in the beginning once and for all:

$$\text{GLODEN} \leftarrow p\text{-adic form of } \prod_{i=1}^{n-1} b_{i, i+1}. \quad (22)$$

This results in considerable reduction in computation and complexity.

*Note:* In fact, we require here the computation of

$$\overline{\text{GLODEN}} \leftarrow \prod_{i=1}^{n-1} b_{i, i+1}$$

only, although in the numerical example we actually compute GLODEN and then  $\overline{\text{GLODEN}}$  as a matter of illustration.

### 6. Numerical example

Compute an exact symmetrizer for the matrix (lower Hessenberg)

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Since the elements of  $B$  are integers, it is clear from the algorithm that

$$\text{GLODEN} = p\text{-adic form of } \prod_{i=1}^{n-1} b_{i, i+1}$$

and that during  $p$ -adic multiplication/division it undergoes no modification.

*Note* : If the elements of  $B$  are (nonintegral) rational numbers then multiply  $B$  by the scalar matrix whose  $(i, i)$ -th elements are the LCM of the denominators of all the rational elements. This results in an integer matrix. Compute a symmetrizer of this integer matrix. Multiply this symmetrizer by the scalar matrix to obtain a symmetrizer of  $B$ .

$$\text{Let max. col. sum} = \max_j \sum_{i=1}^n |b_{ij}| = 6.$$

$$n.(\text{max. col. sum})^{n-1} = 3 \times 6^2 = 108.$$

Therefore  $p = 5$  and  $r = 7$  will serve our purpose since

$$\sqrt{((p^r - 1)/2)} = \sqrt{((5^7 - 1)/2)} \geq 108.$$

Choose  $x_3 = (1 \ 0 \ 0)$ .

(i) *Encoding the input data into  $p$ -adic form* :  $B$ , in  $p$ -adic form, is

$$B' = \begin{bmatrix} \cdot 2000000 \text{ E0} & \cdot 1000000 \text{ E0} & \cdot 0000000 \text{ E0} \\ \cdot 3000000 \text{ E0} & \cdot 1000000 \text{ E0} & \cdot 2000000 \text{ E0} \\ \cdot 1000000 \text{ E0} & \cdot 2000000 \text{ E0} & \cdot 1000000 \text{ E0} \end{bmatrix}$$

$x_3$ , in  $p$ -adic form, is

$$x'_3 = (\cdot 1000000 \text{ E0} \quad \cdot 0000000 \text{ E0} \quad \cdot 0000000 \text{ E0})$$

The global denominator GLODEN is given by

$$\begin{aligned} \text{GLODEN} &= p\text{-adic form of } \prod_{i=1}^{n-1} b_{i, i+1} \\ &= p\text{-adic form of } 1 \times 2 = \cdot 2000000. \end{aligned}$$

*Note* : The global denominator is always a positive number as we attach the negative sign (if any) to the numerator only. Thus GLODEN has always an exponent 0 (viz., E0) and we do not write E0 at all.

(ii) *Executing the Datta Algorithm using  $p$ -adic arithmetic* :

$$\begin{aligned} x'_2 &= \frac{1}{b'_{23}} (x'_3 B' - b'_{33} x'_3) \\ &= (\cdot 3222222 \text{ E0} \quad \cdot 3222222 \text{ E0} \quad \cdot 0000000 \text{ E0}) \end{aligned}$$

$$\begin{aligned} x'_1 &= \frac{1}{b'_{12}} (x'_2 B' - b'_{22} x'_2 - b'_{32} x'_3) \\ &= (\cdot 0000000 \text{ E0} \quad \cdot 3222222 \text{ E0} \quad \cdot 1000000 \text{ E0}). \end{aligned}$$

(iii) Converting  $p$ -adic output data to rational form : Converting the  $p$ -adic output data  $x'_1, x'_2, x'_3$  into rational form we obtain  $x_1, x_2, x_3$  as follows :

$$x'_{11} = \cdot 0000000 \text{ E0} = m_a E e_a, \quad \text{GLODEN} = \cdot 2000000$$

$$m_\gamma = m_a \times \text{GLODEN} = \cdot 0000000 \times \cdot 2000000 = \cdot 0000000$$

$$\overline{\text{GLODEN}} = 2 \times 5^0 + 0 \times 5^1 + 0 \times 5^2 + 0 \times 5^3 + 0 \times 5^4 + 0 \times 5^5 + 0 \times 5^6 \\ = 2.$$

$$v(\overline{\text{GLODEN}}) = 2 \text{ since } \overline{\text{GLODEN}} \leq (p^f - 1)/2 = (5^7 - 1)/2$$

(This is always the case).

$$I(m_\gamma) = 0 \times 5^5 + 0 \times 5^4 + 0 \times 5^3 + \dots + 0 \times 5^0 = 0$$

$$v(I(m_\gamma)) = 0 \text{ since } m_\gamma \leq (p^f - 1)/2 = (5^7 - 1)/2$$

Since  $e_a = 0$ ,  $p^{e_a} = 5^0 = 1$ .

Therefore,

$$x_{11} = \frac{1}{p^{e_a}} \cdot \frac{v(I(m_\gamma))}{v(\overline{\text{GLODEN}})} = \frac{1}{1} \cdot \frac{0}{2} = 0.$$

$$x'_{12} = \cdot 3222222 \text{ E0} = m_a E e_a, \quad \text{GLODEN} = \cdot 2000000$$

$$m_\gamma = m_a \times \text{GLODEN} = \cdot 3222222 \times \cdot 2000000 = \cdot 1000000$$

As before,  $v(\overline{\text{GLODEN}}) = 2$ .

$$I(m_\gamma) = 1 \times 5^0 + 0 \times 5^1 + \dots + 0 \times 5^6 = 1$$

$$v(I(m_\gamma)) = 1 \text{ since } m_\gamma \leq (p^f - 1)/2 = (5^7 - 1)/2$$

Since  $e_a = 0$ ,  $p^{e_a} = 5^0 = 1$

$$x_{12} = \frac{1}{p^{e_a}} \cdot \frac{v(I(m_\gamma))}{v(\overline{\text{GLODEN}})} = \frac{1}{1} \cdot \frac{1}{2} = \frac{1}{2}.$$

Similarly we can obtain  $x_{13}$ , and  $x_2, x_3$ . Hence, the symmetrizer

$$X = (x_1 \ x_2 \ x_3) = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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