Stress analysis of composites through photo-anisotropic elasticity

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Abstract

Various aspects of the photo-anisotropic elasticity method are systematically analysed and its potential for stress analysis of composites is examined. Various theories and proposals for interpreting the photoelastic response of the birefringent composites have been comprehensively studied and critically evaluated in order to establish a firm basis for the method. The application potential of the method is demonstrated and the shortcomings are exposed with a view to improve the method with further investigations. An exhaustive bibliography, compiled on chronological order, is presented.

Key words : Photo-anisotropic elasticity, photoelasticity, composite materials, stress analysis.

1. Introduction

Pih and Knight[1] initiated the use of transparent birefringent composites with anisotropic elastic and optical properties. These orthotropic materials are for model studies using transmission photo-elasticity. Recently many investigations have been conducted in this direction for fabrication of suitable model materials and to interpret their photo-elastic response.

The birefringent composites are manufactured using glass fibres embedded in a birefringent resin matrix having a matching refractive index. The elastic and birefringent properties of the composite materials depend on the properties of the individual constituents.

Pih and Knight[1] developed a stress-optic law based on a stress proportioning technique. Later Sampson[5] formulated a stress-optic law which hypothesised the concept of Mohr-circle of birefringence. In this concept three photoelastic constants are involved

129

to photoelastically characterise these new materials. Bert[14] applied Bhagavantam's theory of photoelasticity for an orthotropic crystalline system to develop a general theory of photo-orthotropic elasticity and showed that the concept of Mohr's circle of birefringence as proposed by Sampson is a direct result of tensorial nature of birefringence.

Pipes and Rose[24] have shown that a single strain optic coefficient coupled with the four independent material constants are sufficient for prediction of the optical response of a birefringent anisotropic (orthotropic) material. Similar conclusion has earlier been drawn by Netrebko *et al*[5].

Dally and Prabhakaran[10] suggested a method for fabricating transparent birefringent models and employed a stress-strain model to predict the three fundamental photoelastic constants based upon the properties of the constituents.

Jan Cernosek[28] used the theory of unitary system of retarders to verify the existing theories of photo-orthotropic elasticity (phenomenological theory and stress-proportioning concept) and discussed the effect of heterogeneity of the material on photoelastic response.

Further investigations have been conducted to verify the method suggested for interpreting the photoelastic response of birefringent composites[32, 33, 38] to predict the optical characteristics from component properties[10, 26, 27, 32] and towards application of the method for practical problems[13, 15, 19]. Some of these investigations are of exploratory nature and conclusions are suggestive but incomplete in realising the phenomena.

Various theories and proposals for interpreting the photoelastic response of the birefringent composites have been critically evaluated to establish a firm basis for the method.

2. Anisotropic theory

The composite materials are generally anisotropic and the basic unit considered is an orthotropic lamina generally fabricated out of a matrix unidirectionally reinforced with relatively stiff and strong fibres. In a macro-mechanical analysis the theory of elasticity for orthotropic materials will hold good.

The basic equations as applied to two-dimensional orthotropic composites are:

(i) Equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{1}$$

(ii) The strain compatibility equation

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \, \partial y} \tag{2}$$

For an orthotropic sheet in which applied stresses are not in general coincident with the axes of orthotropy, the stress-strain relations (Hooke's law) can be written as

$$\begin{bmatrix} e_x \\ e_y \\ e_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{3}$$

where S_{ij} are the elastic compliances.

With the help of eqns. (3) and (1) we can reformulate the compatibility equation[1] into the form

$$(2S_{12} + S_{66})\frac{\partial^2 \sigma_x}{\partial x^2} + S_{11}\frac{\partial^2 \sigma_x}{\partial y^2} + S_{22}\frac{\partial^2 \sigma_y}{\partial x^2} - 2S_{16}\frac{\partial^2 \sigma_x}{\partial x \partial y} - 2S_{26}\frac{\partial^2 \sigma_y}{\partial x \partial y} = 0. \tag{4}$$

By introducing the stress function Φ (Airy's), where

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \tag{5}$$

into eqn. (4) we obtain

$$S_{11}\frac{\partial^4 \Phi}{\partial y^4} + (2S_{12} + S_{66})\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + S_{22}\frac{\partial^4 \Phi}{\partial x^4}$$
$$- 2S_{16}\frac{\partial^4 \Phi}{\partial x \partial y^3} - 2S_{26}\frac{\partial^4 \Phi}{\partial x^3 \partial y} = 0 \tag{6}$$

For the special case, in which stresses act in directions parallel or perpendicular to the orthotropic axes ($S_{16} = S_{26} = 0$), eqn. (6) becomes

$$S_{11}\frac{\partial^4 \Phi}{\partial y^4} + (S_{66} + 2S_{12})\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + S_{22}\frac{\partial^4 \Phi}{\partial x^4} = 0. \tag{7}$$

In an isotropic solid ($S_{11} = S_{22} = 1/2\ (S_{66} + 2S_{12})$) eqn. (7) takes the form

$$\frac{\partial^4 \Phi}{\partial y^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial x^4} = 0. \tag{8}$$

The solution of any one of the eqns. (6), (7) or (8), subject to appropriate boundary conditions, represents a typical plane stress problem.

It will be observed by inspection that the solution to the isotropic problem (eqn. (8)) depends on the geometry of the structure, whereas solution to either of the orthotropic eqns. (6) and (7) depends on both geometry and relation between the elastic constants.

2.1. Orthotropic constants

For the case where stresses act in directions parallel or perpendicular to the orthotropic axes which usually coincide with the parallel and perpendicular to fibre directions in

unidirectionally reinforced composites, the stress-strain relations can be written with respect to the orthotropic co-ordinates (Fig. 1).

$$\begin{bmatrix} e_l \\ e_t \\ e_{lt} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & O \\ S_{12} & S_{22} & O \\ O & O & S_{66} \end{bmatrix} \cdot \begin{bmatrix} l \\ \sigma_t \\ \tau_{lt} \end{bmatrix} \tag{9}$$

The compliance terms can be written in terms of engineering symbols as

$$S_{11} = \frac{1}{E_l}$$

$$S_{12} = \frac{-v_{lt}}{E_l} = -\frac{v_{tl}}{E_t}$$

$$S_{22} = \frac{1}{E_t}$$

$$S_{66} = \frac{1}{G_{lt}} \tag{10}$$

or we can write

$$e_l = \frac{\sigma_l}{E_l} - v_{tl}\frac{\sigma_t}{E_t}$$

$$e_t = \frac{\sigma_t}{E_t} - v_{lt}\frac{\sigma_l}{E_l}$$

$$e_{lt} = \frac{\tau_{lt}}{G_{lt}}. \tag{11}$$

Due to symmetry of the stress-strain matrix we have

$$v_{tl}\, E_l = v_{lt}\, E_t \tag{12}$$

XY—Arbitrary co-ordinates
LT—Orthotropic co-ordinates
L—Parallel to fibre orientation
T—Transverse to fibre orientation
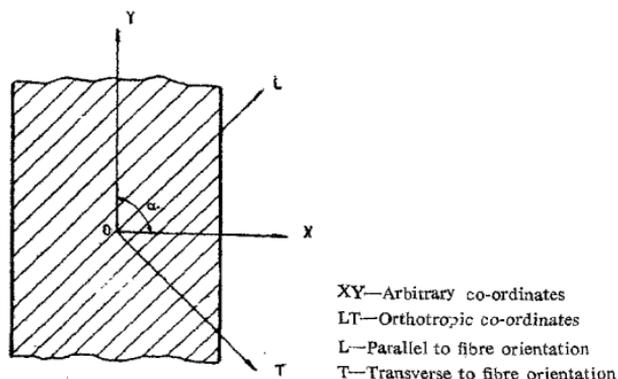
FIG. 1. Orthotropic lamina and co-ordinate system.

The compatibility eqn. (7) can now be written as

$$\left(\frac{\partial^2}{\partial l^2} + K_1^2 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial l^2} + K_2^2 \frac{\partial^2}{\partial t^2}\right) \Phi = 0 \tag{13}$$

or in terms of stresses as

$$\left(\frac{\partial^2}{\partial l^2} + K_1^2 \frac{\partial^2}{\partial t^2}\right) \left(\sigma_t + \frac{2}{t}\sigma_l\right) = 0 \tag{14}$$

where $\quad K_1^2 + K_2^2 = (S_{66} + 2S_{12})/S_{22} = \dfrac{E_t}{G_{lt}} - 2\,v_{tl} = A$ \hfill (15)

$$K_1^2\,K_2^2 = S_{11}/S_{22} = \frac{E_t}{E_l} = B \tag{16}$$

K_1 and K_2 are called orthotropic constants which characterise the orthotropic medium. The stress distribution in such an orthotropic medium depends on these constants. It may be noted that the constants depend on the ratio of elastic modulii.

2.2. *Graphic representation of anisotropy*

The variation of elastic constants with changes of direction can be represented graphically by several methods. One method is to plot distances proportional to elastic modulii in each direction (Fig. 2).

$$E_a = E_l \cos^2 \alpha + 2G_{lt} \sin \alpha \cos\alpha + E_t \sin^2 \alpha. \tag{17}$$

Another way to represent the variation is by employing the direction curve given by the following relation (Fig. 3):

$$Bx^4 + Ax^2\,y^2 + y^4 = B \tag{18}$$

where A and B are given by eqns. (15) and (16). At times the coefficients of strain are plotted with respect to direction to obtain the curve of coefficients of strain (Fig. 4).

$$r = KS_{11}' \tag{19}$$

where $K =$ scale factor

and $S_{11}' = S_{11} \cos^2 \alpha + 2S_{12} \sin \alpha \cos \alpha + S_{22} \sin^2 \alpha.$ \hfill (20)

3. Photoelastic response of composites

3.1. *Circle of birefringence*

Consider the isotropic stress-optic law

$$\sigma_1 - \sigma_2 = Nf_\sigma \tag{21}$$
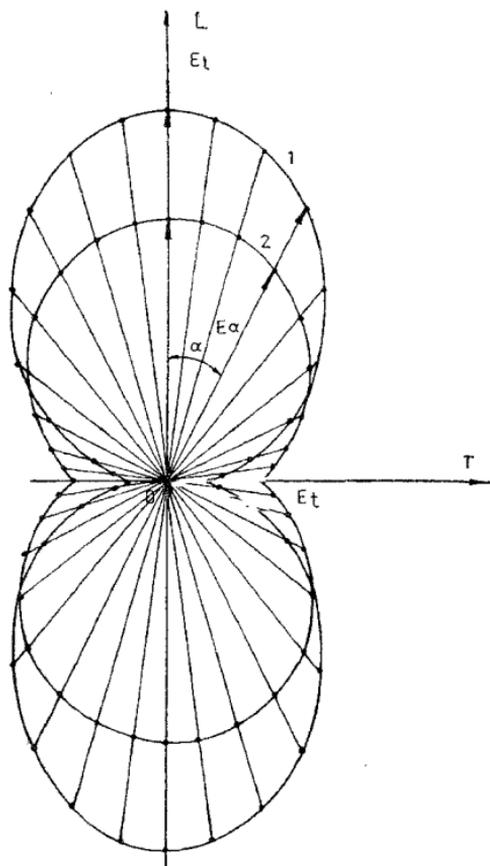
Curve 1. Boron-epoxy composite
$E_l = 30 \times 10^5 \text{—} \text{kg/cm}^2$
$E_t = 3 \times 10^5 \text{kg/cm}^2$
$G_{lt} = 1 \cdot 2 \times 10^5 \text{kg/cm}^2$
$\nu_{lt} = 0 \cdot 25$

Curve 2. Glass-epoxy composite
$E_l = 7 \times 10^5 \text{kg/cm}^2$
$E_t = 2 \cdot 5 \times 10^5 \text{kg/cm}^2$
$G_{lt} = 0 \cdot 8 \times 10^5 \text{kg/cm}^2$
$\nu_{lt} = 0 \cdot 25$

Fig. 2. Variation of elastic modulus with direction for typical composites.

where N is fringe order per unit thickness and f_σ is the stress-optic coefficient. But

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left\{ \left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right\}} \tag{22}$$

when combined, eqns. (21) and (22) yield

$$N^2 = \left(\frac{\sigma_x}{f_\sigma} - \frac{\sigma_y}{f_\sigma} \right)^2 + \left(\frac{2\tau_{xy}}{f_\sigma} \right)^2 \tag{23}$$
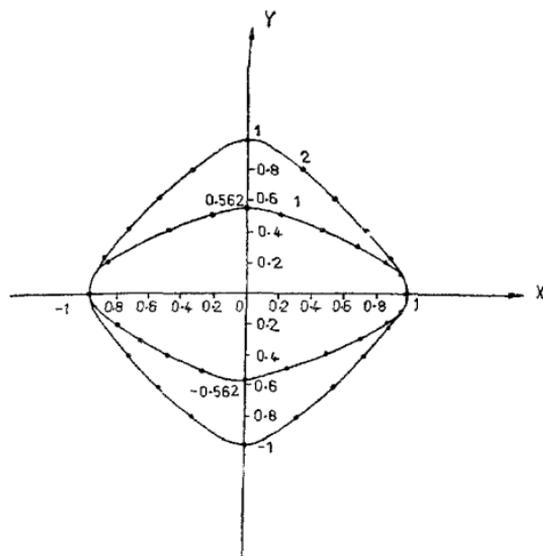
Curve 1. Boron-epoxy composite

$E_l = 30 \times 10^5\,\text{kg/cm}^2$
$E_t = 3 \times 10^5\,\text{kg/cm}^2$
$_\mu = 1 \cdot 2 \times 10^5\,\text{kg/cm}^2$
$n_{lt} = 0 \cdot 25$
$A = 2$
$B = 0 \cdot 1$

Curve 2. Glass-epoxy composite

$E_l = 7 \times 10^5\,\text{kg/cm}^2$
$E_t = 2 \cdot 5 \times 10^5\,\text{kg/cm}^2$
$G_{lt} = 0 \cdot 8 \times 10^5\,\text{kg/cm}^2$
$\nu_{lt} = 0 \cdot 25$
$A = 3 \cdot 125$
$B = 0 \cdot 56$

FIG. 3. Direction curves for typical composites.

For the anisotropic materials the stress-optic law should have the same form as eqn. (23) but possess three independent stress-optic coefficients f_x, f_y and f_{xy}, then

$$N^2 = \left(\frac{\sigma_x}{f_x} - \frac{\sigma_y}{f_y}\right)^2 + \left(\frac{2\tau_{xy}}{f_{xy}}\right)^2 \tag{24}$$

Here it can be seen that the birefringent components contributed by each component of plane stress are combined according to a circle of birefringence concept. The components of birefringence are defined as

$$N_x = \frac{\sigma_x}{f_x},\ N_y = \frac{\sigma_y}{f_y},\ N_{xy} = \frac{\tau_{xy}}{f_{xy}}. \tag{25}$$
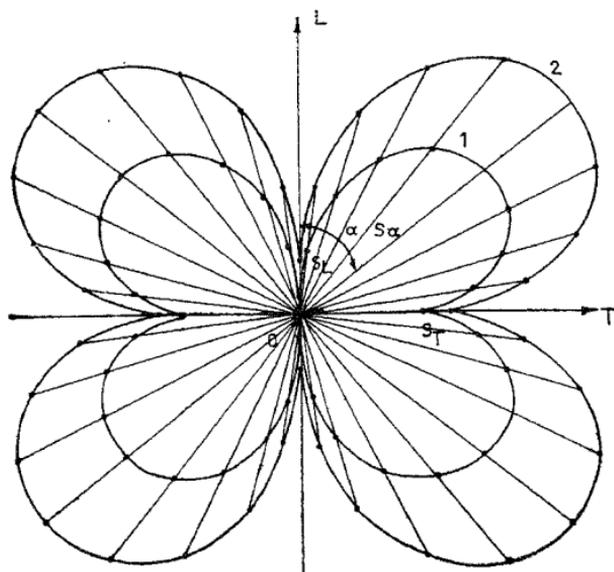
Curve 1. Boron-epoxy composite

$S_{11} = 0.33 \times 10^{-6} \, cm^2/kg$

$S_{22} = 3.3 \times 10^{-6} \, cm^2/kg$

$S_{66} = 8.3 \times 10^{-6} \, cm^2/kg$

$S_{12} = -0.083 \times 10^{-6} \, cm^2/kg$

Curve 2. Glass-epoxy composite

$S_{11} = 1.4 \times 10^{-6} \, cm^2/kg$

$S_{22} = 4 \times 10^{-6} \, cm^2/kg$

$S_{66} = 12.5 \times 10^{-6} \, cm^2/kg$

$S_{12} = -0.35 \times 10^{-6} \, cm^2/kg$

Fig. 4. Curves of coefficients of strain for typical composites.

Then the circle of birefringence can be developed as (Fig. 5)

$$N^2 = (N_x - N_y)^2 + (2N_{xy})^2 \tag{26}$$

$$N_{1,2} = \frac{N_x + N_y}{2} \pm N/2$$

$$N_x - N_y = N \cos 2\theta'$$

$$N_{xy} = \frac{N}{2} \sin 2\theta'. \tag{27}$$
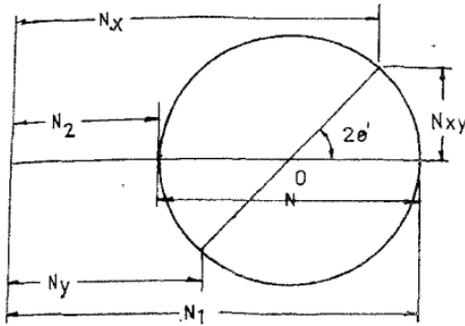
$$N_{1,2} = (N_x + N_y)/2 \pm N/2$$
$$N_x - N_y = N \cos 2\theta'$$
$$N_{xy} = \frac{N}{2} \sin 2\theta'$$

FIG. 5. Circle of birefringence.

Now, from the isochromatic fringe order and isolinic parameter we can obtain the following relationships:

$$\frac{\sigma_x}{f_x} - \frac{\sigma_y}{f_y} = N \cos 2\theta' \tag{28}$$

$$\tau_{xy} = N \frac{f_{xy}}{2} \sin 2\theta'. \tag{29}$$

3.2. Strain-optic law

The classical strain-optic law for isotropic materials can be expressed as

$$e_1 - e_2 = N f_\varepsilon. \tag{30}$$

The principal strains, expressed in terms of arbitrary co-ordinate system, x, y, are

$$e_1, e_2 = \frac{e_x + e_y}{2} \pm \sqrt{\left\{\left(\frac{e_x - e_y}{2}\right)^2 + \left(\frac{e_{xy}}{2}\right)^2\right\}}. \tag{31}$$

Hence
$$e_1 - e_2 = \sqrt{\{(e_x - e_y)^2 + (e_{xy})^2\}} \tag{32}$$

i.e.,
$$N = \frac{1}{f_\varepsilon} \sqrt{\{(e_x - e_y)^2 + e_{xy}^2\}} \tag{33}$$

or
$$N^2 = \left(\frac{e_x - e_y}{f_\varepsilon}\right)^2 + \left(\frac{e_{xy}}{f_\varepsilon}\right)^2. \tag{34}$$

The stress-strain relationships for an anisotropic material, subjected to a state of plane stress is given by eqn. (3). When the medium possesses a plane of elastic symmetry S_{16} and S_{26} the shear coupling compliance terms vanish, then

$$(e_x - e_y)^2 = [(S_{11} - S_{12}) \sigma_x - (S_{22} - S_{12}) \sigma_y]^2 \tag{35}$$

$$(e_{xy})^2 = (S_{66} \tau_{xy})^2 \tag{36}$$

when combined, eqns. (34), (35) and (36) yield the strain-optic law

$$N^2 = \left[\frac{(S_{11} - S_{12})}{f_\varepsilon} \sigma_x - \frac{(S_{22} - S_{12})}{f_\varepsilon} \sigma_y\right]^2 + \left[\frac{S_{66}}{f_\varepsilon} \tau_{xy}\right]^2. \tag{37}$$

Hence there exist definite relationships between the stress optic coefficients and the strain-optic coefficients which can be obtained by comparing coefficients of stress components from eqns. (37) and (24).

$$f_s = \frac{f_e}{(S_{11} - S_{12})} = \beta_1 f$$

$$f_y = \frac{f_e}{(S_{22} - S_{12})} = \beta_2 f_e$$

$$f_{sy} = \frac{f_e}{S_{66}} = \beta_3 f_e \tag{38}$$

The compliance terms S_{ii} can be expressed in terms of engineering properties of the material by eqn. (10). Then

$$\beta_1 = \frac{E_l}{1 + \nu_{lt}}$$

$$\beta_2 = \frac{E_t}{1 + \nu_{lt}}$$

$$\beta_3 = G_{lt} \tag{39}$$

4. Interpretation of photoelastic data

The photoelastic data will provide a relation between normal stresses [eqns. (28) and (29)].

$$\sigma_s - K_1 \sigma_y = N f_e \cos 2\theta' = N f_e \beta_1 \cos 2\theta' \tag{40}$$

$$K_2 \tau_{sy} = \frac{N}{2} f_{sy} \sin 2\theta' = \frac{N}{2} f_e \beta_1 \sin 2\theta' \tag{41}$$

where θ' is the optical isoclinic and

$$K_1 = \frac{\beta_2}{\beta_1} = \frac{E_t}{E_l} \frac{1 + \nu_{lt}}{1 + \nu_{tl}} = B \frac{1 + \nu_{lt}}{1 + \nu_{tl}}$$

$$K_2 = \frac{\beta_3}{\beta_1} = \frac{G_{lt}}{E_l} (1 + \nu_{tl}) = \frac{\lambda}{2}. \tag{42}$$

For isotropic materials the strain fringe value f_e and stress fringe value f_σ are related as

$$f_\sigma = \frac{E}{1 + \nu} f_e.$$

Similar relations for orthotropic cases are $\tag{43}$

$$f_s = f_l = \frac{E_l}{1 + \nu_{lt}} f_e$$

$$f_y = f_t = \frac{E_t}{1 + \nu_{tl}} f_e$$

$$f_{sy} = f_{lt} = G_{lt} f_e \tag{44}$$

In effect, one can appreciate the fact that the isochromatics in an orthotropic medium represent the contours of maximum shear strain and the isoclinics represent the contours of equal principal strain angles,

4.1. *Isochromatics and isoclinics*

An orthotropic square plate model was fabricated by embedding transparent glass fibres (*E*-glass) unidirectionally in an epoxy matrix of matching refractive index (Araldite CY 230 + Hardner HY 951 + Dibutyl phthalate, $100 + 10 + 10$). The fibre volume fraction was estimated by separating the fibres by burning off the resin at an elevated temperature (around 500° C) in a furnace. The fibre volume fraction in this case was $5\cdot28\%$.

The orthotropic constants were determined from the elastic constants evaluated using tensile specimens. These constants for the two cases studied are as follows :

Fibres vertical $\quad K_1^2 = 2\cdot25$

$\qquad\qquad\qquad K_2^2 = 0\cdot55$

Fibres horizontal $\quad K_1^2 = 1\cdot32$

$\qquad\qquad\qquad K_2^2 = 0\cdot487.$

The models were loaded under partial edge compression (in plane) and the corresponding isochromatic patterns are presented in Fig. 6. Typical isoclinic patterns are presented in Fig. 7.

The quality of the fringes improves when the fibres are well dispersed in the matrix and entrapped air bubbles are not present. For clear fringes and transparency of the model, the refractive indices of both the matrix and the fibres should be the same.

(a)   (b)
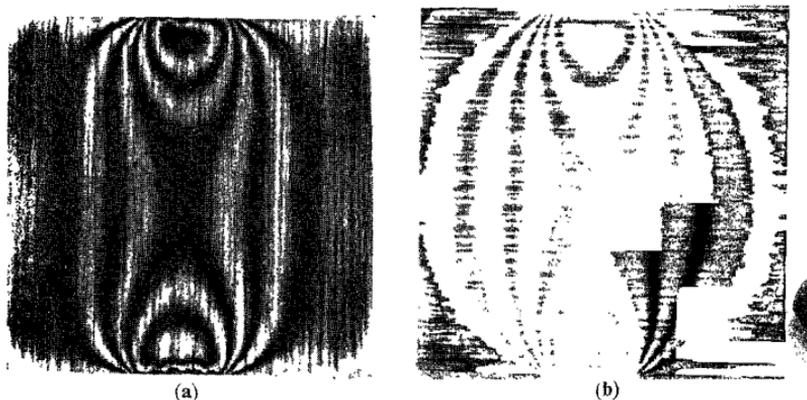
FIG. 6. Typical isochromatic patterns (light field) for an orthotropic square plate subjected to in-plane loading (*a*. parallel to fibre direction, *b*. Normal to fibre direction).
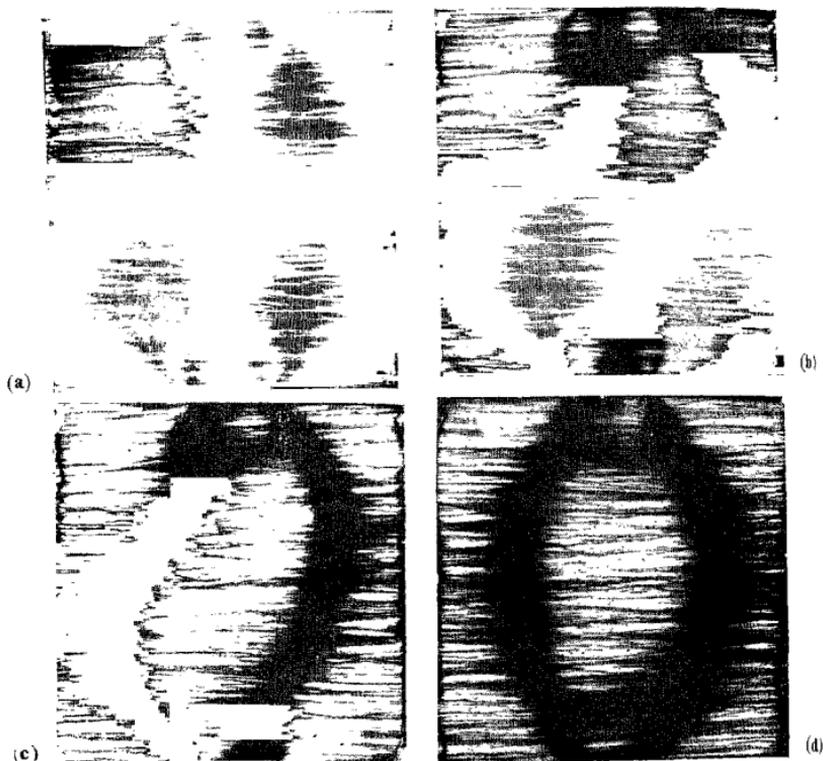
FIG. 7. Typical isoclinic pattern for a square plate subjected to in-plane loading (a. $0°$, b. $15°$, c. $30°$, d. $45°$).

The residual birefringence introduced due to shrinkage of resin during curing is to be accounted in analysing the fringe data. The zero order fringes indicate either zero stress region or points where isotropic strains (not isotropic stress) exist.

4.2. *Separation of stresses*

From the photoelastic data one obtains a linear relationship between the normal stresses and shear stress at each point. The separate values of normal stresses can be

obtained by different methods. The differential equations of equilibrium can be used for this purpose. The derivative of shear stress can be approximated as (Fig. 8)

$$\frac{\partial \tau_{xy}}{\partial y} = \frac{\tau_{xyB} - \tau_{xyA}}{\triangle y} \tag{45}$$

and then

$$\sigma_x = \sigma_{x,0} - \int_0^x \frac{\partial \tau_{xy}}{\partial y} \, dx. \tag{46}$$

The numerical procedure is similar to that used in isotropic case. However, it may be noted that in general $\sigma_{x,0} = 0$ but rather

$$\sigma_{x,0} = P_0 \cos 2\theta_0 \tag{47}$$

where $P_0 =$ tangent stress

and $\theta_0 =$ physical isoclinic.

After obtaining σ_x,

$$\sigma_y = \frac{f_y}{f_x} (\sigma_x - f_x N \cos 2\theta). \tag{48}$$

This constitutes complete solution of the problem.

The equation of equilibrium for an element in cylindrical co-ordinates is

$$\frac{\partial \sigma_r}{\partial_r} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \tag{49}$$
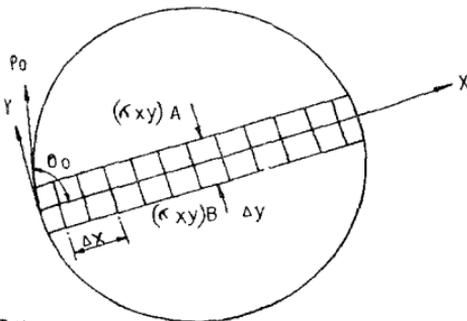
$\tau_{x,\theta} = P_0 \cos 2\theta_0$

FIG. 8. Typical integration grid for shear difference method.

Assuming no body forces, the stress-optic law is

$$\frac{N}{d} = \frac{\sigma_\theta}{f_\theta} - \frac{\sigma_r}{f_r}. \tag{50}$$

By substituting $(\sigma_r - \sigma_\theta)$ obtained from eqn. (49), eqn. (50) becomes

$$\frac{N}{d} = \sigma_r\left(\frac{1}{f_\theta} - \frac{1}{f_r}\right) + \frac{1}{f_\theta}\left(r\frac{\partial\sigma_r}{\partial_r}\right). \tag{51}$$

Equation (51) can be used in its finite difference form to evaluate σ_r along a radial line.

Another method is by employing the numerical solution of orthotropic compatibility equation expressed in terms of stresses

$$\left(\frac{\partial^2}{\partial x^2} + K_1^2\frac{\partial^2}{\partial y^2}\right)S = 0 \tag{52}$$

where $S = \sigma_y + K_2^2\sigma_x$ \hfill (53)

and K_1 and K_2 are the orthotropic constants. Equation (52) can be put in finite difference form using the standard procedure. The recurring relation for S at any interior mesh point 0 (i, j) becomes

$$S_{i,j} = S_{i+1,j} + S_{i-1,j} + (S_{i,j+1} + S_{i,j-1})K_1^2/2(1 + K_1^2). \tag{54}$$

Along the free boundaries, the photoelastic data give the value of the principal stresses. On loaded boundaries, one of the principal stresses is known; hence, from the photoelastic data the stress components can be calculated. After obtaining boundary values, eqn. (56) can be used to calculate the values of S in the 'interior' with the help of an iteration programme.

Then, the values of σ_x and σ_y can be calculated knowing the value of S and the photoelastic data.

$$\sigma_x = \frac{S + Nf_y\cos 2\theta'}{K_2^2 + K} \tag{55}$$

and $\sigma_y = S - K_2^2\sigma_x$ \hfill (56)

where $K = f_x/f_y$.

The shear stress is given by

$$\tau_{xy} = \frac{Nf_{xy}}{2}\sin 2\theta'. \tag{57}$$

5. Prediction of birefringence of composites

The optical response in terms of fringe order due to each constituent is algebraically summed up to give the total retardation as

$$N = N_f + N_m \tag{58}$$

(where f and m represent fibre and matrix respectively).

This algebraic summing neglects any influence of rotation of principal stresses as the light propagates through the heterogeneous composite. In the simplest case the photo-elastic constants in orthotropic case can be represented by

$$f_t = \frac{\sigma_t h}{N}, \quad f_t = \frac{\sigma_t h}{N},$$

$$f_{tt} = \frac{2\tau_{tt} \cdot h}{N}. \tag{59}$$

Defining in this manner, the photoelastic constants f_t, f_t and f_{tt} are analogous to the elastic constants E_t, E_t and G_{tt} and for a unidirectionally-reinforced composite. The law of mixtures formulae for computing elastic constants can be analogously derived for the photoelastic constants. Then

$$f_t = \frac{V_f + (E_m/E_f) V_m f_f f_m}{V_f f_m + V_m f_f (E_m/E_f)} \tag{60}$$

$$f_t = (f_f f_m)/(V_f f_m + V_m f_f) \tag{61}$$

$$f_{tt} = f_f f_m (V_f f_m + V_m f_f). \tag{62}$$

The above relations can be modified by considering more realistic (also more complex) stress-strain models[26], [34].

6. The isoclinic paradox

The anisotropic elasticity theory says that the principal stresses and strains are in general case non-coincident except at certain points of symmetry in an anisotropic elastic media. This deviation can be obtained exactly in theory. But in photo-anisotropic elasticity, the problem involved is in the interpretation of the optical isoclinic and in determining the exact principal stress or strain directions from the measured parameter. The available literature in this regard is suggestive but a unique conclusion in this regard is absent.

Pih and Knight[1] observed that the optical isoclinic angle differed considerably from the physical isoclinic angle and Sampson[5] suggested that the optical isoclinic angle is one half of the angle between the fibre direction and the direction of the principal component of birefringence as determined by Mohr-circle. Prabhakaran[32] investigated the isoclinics in orthotropic photoelastic models and suggested that they can be interpreted by circle of birefringence concept and as an approximation the principal strain angles are closer compared to the principal stress angles. Jan Cernosek[28] showed that the heterogeneous composite material is a simple linear retarder and no rotation is present. He suggests that the stress-optical constant and parameter of optical isoclinic can be accurately predicted even if the residual birefringence is present in the unloaded specimen. Pipes and Rose[24] hypothesised the isoclinic angle to be equal to the principal strain direction of equivalent homogeneous anisotropic medium and expressed the doubt that the actual heterogeneity of the materials might be expected to alter the physics of the isoclinic parameter.

Prabhakaran[34] employed a stress strain model to determine the principal directions in the constituents. He observed that the principal strain directions for the composites are significantly different from the principal stress directions. Also the principal directions for the constituents are different from each other and are both different from the principal stress or strain directions for the composite. He also observed that the principal strain directions for the matrix are close to that of the composite as a whole. However, no proper explanation is given to validate these observations. The non-coincidence of principal directions in constituents in a way suggests lack of compatibility and inadequacy of the stress-strain model to predict the gross behaviour.

6.1. *Correction for isoclinic angle*

The basic principle of isotropic photoelasticity theory is that plane-polarized light vector is resolved into two components which are parallel to principal stress or strain axes at each point in a birefringent medium. Extinction fringes called isoclinics are produced when the principal axes are oriented parallel or perpendicular to the polarizer. The strain optic law suggests that the light vector is resolved into components which are parallel to the principal strain axes. Unlike isotropic materials, the anisotropy of these materials yield non-coincident principal stress and strain directions. The actual heterogeneity of the material might be expected to alter the physics of the isoclinic parameter. The principal strain direction is given by

$$\tan 2\theta_e = \frac{e_{tt}}{e_t - e_t} = \frac{2\tau_{lt}}{\sigma_t} \; \frac{E_t}{2G_{tt}(1 + \nu_{tt})}. \tag{63}$$

This is the plane on which there is no shear strain but not necessarily zero shear stress.

The direction of the planes of principal stress (*i.e.*, planes on which shear stress but not necessarily shear strain is zero) is then given by

$$\tan 2\theta_\sigma = \frac{2\tau_{lt}}{\sigma_t}. \tag{64}$$

The angles θ_e and θ_σ are angles subtended between the plane which has the fibre axis, as normal and the plane of principal strain or principal stress respectively.

For isotropic materials, $E = 2G (1 + \nu)$ and then eqns. (65) and (66) are identical or the planes of principal stress and strain are coincident.

From eqns. (65) and (66) we may write for the plane case in which $\sigma_t = 0$, that

$$\frac{\tan 2\theta_e}{\tan 2\theta_\sigma} = \frac{t}{2G_{tt}(1 + \nu_{tt})} = \lambda \tag{65}$$

which is an equation describing the difference between the directions of the planes of principal stress and strain in terms of orthotropic elastic constants. Obviously for an isotropic material ($\lambda = 1$) the isoclinics of stress and strain are coincident.

Using eqn. (67) we can obtain the principal stress angles from principal strain direction.

$$\theta_\sigma \simeq \frac{1}{2} \tan^{-1} \left(\frac{\tan 2\theta_e}{\lambda} \right). \tag{66}$$

Now in eqns. (28) and (29) the optical isoclinic parameter may be replaced by the principal stress angles.

$$\theta_\sigma = \frac{1}{2} \tan^{-1} \left(\frac{\tan 2\theta'}{\lambda} \right). \tag{67}$$

The deviation of θ_σ from θ' and its dependence on λ is demonstrated in Fig. 9.

Using the above concept, the shear stresses and a relation between normal stresses are obtained along different sections for the orthotropic square plate model.

$$\tau_{lt} = C_1 F_l \sin 2\theta_\sigma \tag{68}$$

$$\sigma_l - C_2 \sigma_t = N F_l \cos 2\theta_\sigma \tag{69}$$

where $C_1 = F_{lt}/2F_l, \; C_2 = F_t/F_l$

$F_l = \dfrac{f_l}{t}$

$\theta_\sigma = \theta' - \eta_\theta = \dfrac{1}{2} \tan^{-1} \left(\dfrac{\tan 2\theta'}{\lambda} \right).$

For isotropic materials, $G = E/2 (1 + v)$ and hence $\lambda = 1$ and $\theta_\sigma = \theta'$ (observed).
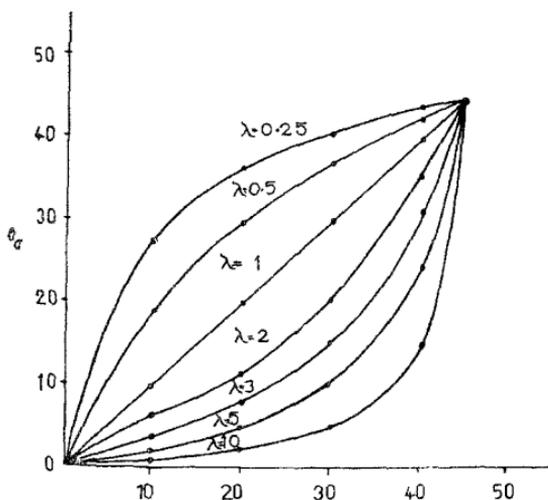
Fig. 9. Deviation of isoclinic angle (θ') from principal stress angle and its dependence on λ.

The stress components σ_x and σ_y are computed along the horizontal axis for the two cases of loading (parallel and perpendicular to fibres). The results are presented in Fig. 10.

. Scope for further research

'or fruitful application of the photo-anisotropic elasticity for stress analysis of composites, certain fundamental difficulties are to be resolved. Further investigations are required to characterise the birefringent composites, both mechanically and optically by minimum experimental measurements, but with high reliability and confidence.

$$\sigma_o = 2\,qa\,/L$$
$$L = 10\,cm$$
$$a = L/10$$

—— · —— Fibres vertical ($K_1^2 = 2\cdot05,\ K_2^2 = 0\cdot55$)
— — · — — Fibres horizontal ($K_1^2 = 1\cdot82,\ K_2^2 = 0\cdot487$)
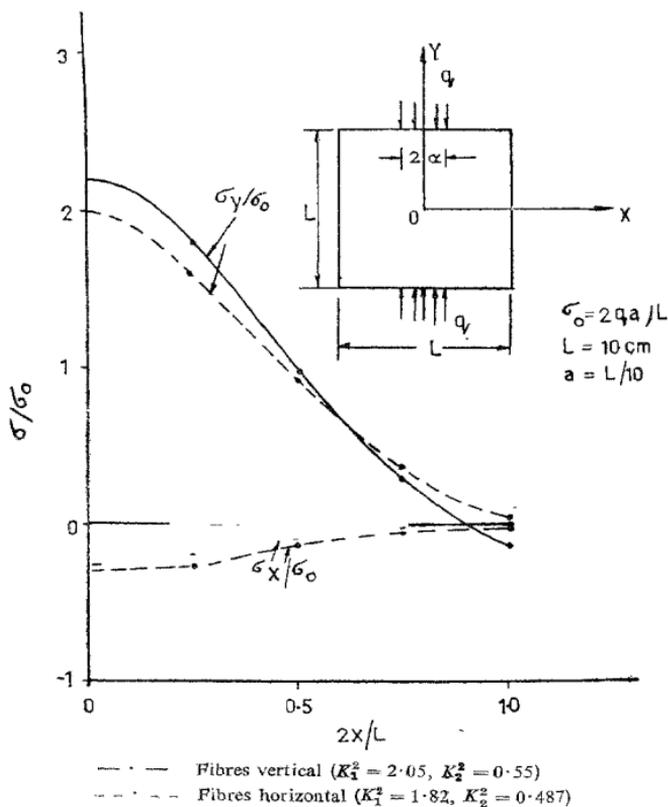
FIG. 10. Distribution of σ_x and σ_y along the axis OX for the orthotropic square plates subjected to partial edge compression.

Most of the advanced composite materials (graphite-epoxy, boron-epoxy, boron-aluminium, etc.) are neither birefringent nor transparent. This necessitates development of suitable model materials and techniques to represent these materials.

The available theories are to be consolidated and improved to uniquely characterise and realise the photoelastic response of the birefringent composites. (These have been partly accomplished in this paper.)

It is also necessary to investigate the lamination effects (coupling effects) in addition to the macroscopic anisotropic effects (properties depending on orientation). This requires a photoelastic model which can be stress-frozen. Some exploratory investigations in this direction are available in literature.

In effect the broad class of problems to be tackled are :

(i) Development of suitable transparent birefringent model material.

(ii) Development of proper calibration techniques for mechanical and optical characterisation of birefringent composites.

(iii) Development of suitable (reliable) method for interpretation of experimental data.

(iv) Definite understanding of the micro-mechanical phenomena and its contribution to macroscopic optical and mechanical properties.

8. Discussion and conclusion

1. The isochromatics and isoclinics from an anisotropic birefringent medium can be interpreted to obtain the strains inducing them.

2. The deformation and the resulting strain is the fundamental cause for change in lattice orientation and spacing and hence the resulting optical phenomena can be interpreted based on a strain-optic rule.

3. Prediction of optical and mechanical properties based on stress proportioning and stress-strain models are useful in designing birefringent anisotropic models and experiments. However, influence of variations within layers, between layers (interfaces) and frequency of composite components are to be accounted for, for more realistic prediction.

4. The coincidence of the circle of birefringence concept and the isotropic strain-optic coefficient suggests a uniqueness in the theory of photo-anisotropic elasticity. This will be beneficial from application point of view.

5. The whole field optical patterns provide an immediate comparison and verification for anisotropic elasticity theory; and it is possible to evaluate the quantitative values of stress components from the optical data.

Notations

E_l, E_t	—	elastic modulii along the orthotropic axes
e_l, e_i, e_{lt}	—	components of strain in orthotropic co-ordinates
e_x, e_y, e_{xy}	—	components of strain in cartesian co-ordinates
f_e	—	strain-optic constant
f_l, f_i, f_{lt}	—	stress-optic constants in orthotropic co-ordinates
f_x, f_y, f_{xy}	—	stress-optic constants in cartesian co-ordinates
f_θ, f_r	—	stress-optic constants in cylindrical co-ordinates
G	—	shear modulus
K_1, K_2	—	orthotropic constants
l, t	—	represent orthotropic axes
N	—	isochromatic fringe order
r, θ, z	—	cylindrical co-ordinate axes
S_{ij}	—	elastic compliances
V	—	volume fraction
x, y	—	cartesian co-ordinate axes
α	—	included angle between orthotropic axis and a general direction
$\beta_1, \beta_2, \beta_3$	—	derived elastic constants
ν	—	Poisson's ratio
δ	—	Partial derivative
Φ	—	Airy's stress function
σ	—	represent normal stress components
τ	—	represent shear stress components
θ'	—	optical (observed) isoclinic angle
θ_σ	—	principal stress angle
θ_e	—	principal strain angle

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