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Derivation of the solution of certain singular integral equations

ALOKNATH CHAKRABARTI
Department of Applied Mathematics, Indian Institute of Science, Bangalore 560 012, India.

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Abstract

It is shown that the application of the Poincare-Bertrand formula when made in a suitable manner produces the solution of certain singular integral equations very quickly, the method of arriving at which, otherwise, is too complicated. Two singular integral equations are considered. One of these equations is with a Cauchy-type kernel and the other is an equation which appears in the wave-guide theory and the theory of dislocations.

A different approach is also made here to solve the singular integral equations of the wave-guide theory and this involves the use of the inversion formula of the Cauchy-type singular integral equation and reduction to a system of Hilbert problems for two unknowns which can be decoupled very easily to obtain the closed form solution of the integral equation at hand.

The methods of the present paper avoid all the complicated approaches of solving the singular integral equation of the wave-guide theory known to-date.

Key words : Singular integral equations, Cauchy-type kernel, Riemann Hilbert problem, wave-guide theory.

1. Introduction

The Poincare-Bertrand formula (PBF) is given by

$$\int_{-1}^{1}\int_{-1}^{1}\frac{g(x,y)}{(x-y)(y-z)}\,dxdy = -\pi^2 g(z,z) + \int_{-1}^{1}\int_{-1}^{1}\frac{g(x,y)}{(x-y)(y-z)}\,dydx,\ [z\in(-1,1)] \tag{1}$$

where singular integrals are understood as their Cauchy principal values and where the function $g(x, y)$ is assumed to satisfy the Hölder condition used by Muskhelishvili[6].

We shall use here the result (1) to solve first the singular integral equation with a Cauchy-type kernel and then a singular integral equation in wave-guide theory on which the work of Williams[10] has come out recently. Williams has investigated two interesting features of the following equation in the wave-guide theory.

$$\int_0^1 \left(\frac{1}{\eta - \sigma} + \frac{p}{\eta + \sigma}\right)\psi(\eta)\, d\eta = h(\sigma)\,(0 < \sigma < 1). \tag{2}$$

These features are :

(i) its relationship to the equation

$$\int_{-1}^{1} \frac{g(y)\,dy}{y - \xi} = \lambda g(\xi) + f(\xi), \quad |\xi| < 1. \tag{3}$$

(ii) The relationship between the solution of eqn. (2) for $h \equiv 1\,(p > -1)$ and the solution of the corresponding homogeneous equation with p replaced by $-p$.

The work on the singular integral eqn. (2), and its generalisation, by Lewin[4], Bueckner[2], Biermann[1] and Peters[8] all involve very complicated complex variable methods associated with the Hilbert problems and the Wiener-Hopf technique, except that Lewin has utilized some simple properties of the operator T defined by

$$T\varphi = \int_{-1}^{1} \frac{\varphi(y)}{y - x}\, dy \tag{4}$$

to solve eqn. (2) in certain situations.

It is the Lewin's form of the solution of eqn. (2) which has suggested utilizing the PBF to such singular integral equations. This will be presented in the next section.

It is also observed that Pennline[7], has tried to give an alternative approach to the problem of solving (2) in the case when $p = 1/2$ and $h = 1$. The approach of Pennline involves the reduction of the problem (2) into a system of Hilbert problems, a closed form solution of which is difficult to obtain and Pennline has obtained the solution only by employing a special technique. What more we demonstrate here in the last section of this paper is that, a reduction of eqn. (2), after a simple transformation and an utility of the inverse operator T^{-1} obtained in section 2, to a simple system of Hilbert problems with constant coefficients for two unknowns, can be achieved relatively cheaply and the final solution of eqn. (2) can be obtained in closed form for any real p.

5. The use of the PBF

As a first use of the PBF to singular integral equations of the first kind, we consider the equation

$$T\phi \equiv \int_{-1}^{1} \frac{\varphi(y)\,dy}{y-x} = f(x), \tag{5}$$

where the functions ϕ and f are assumed to satisfy the Hölder conditions in $(-1, 1)$. Rewrite eqn. (5) as

$$\sqrt{1-x^2} \int_{-1}^{1} \frac{\varphi(y)\,dy}{y-x} = \sqrt{1-x^2}\, f(x). \tag{6}$$

Multiply both sides of (6) by

$$\left(\frac{1-x}{1+x}\right)^{1/2-\beta} \frac{dx}{x-\xi},\ \xi \in (1-1, 1)$$

and integrate with respect to x between $x = -1$ and $+1$, where $0 < \mathrm{Re}(\beta) < 1$. Then interchanging the orders of integration by using the PBF (1), we obtain :

$$-(1-\xi)\left(\frac{1+\xi}{1-\xi}\right)^{\beta} \pi^2 \phi(\xi)$$

$$+ \int_{-1}^{1} \frac{\varphi(y)\,dy}{y-\xi} \left\{ \int_{-1}^{1} (1-x)\left(\frac{1+x}{1-x}\right)^{\beta} \left(\frac{1}{y-x} + \frac{1}{x-\xi}\right) dx \right\}$$

$$= \int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{1/2-\beta} \frac{\sqrt{1-x^2}\, f(x)}{x-\xi}\, dx. \tag{7}$$

If we now use the following result (see Gakhov[3]),

$$\int_{-1}^{1} (1-x)\left(\frac{1+x}{1-x}\right)^{\beta} \frac{dx}{x-\lambda}$$

$$= (1-\lambda)\left[\frac{\pi}{\sin \pi\beta} - \pi \cot \pi\beta \left(\frac{1+\lambda}{1-\lambda}\right)^{\beta}\right] - C(\beta), \quad \text{for } -1 < \lambda < 1 \tag{8}$$

where

$$C(\beta) = \int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{\beta} dx, \tag{9}$$

eqn. (7) takes the following form

$$- (1 - \xi)\left(\frac{1+\xi}{1-\xi}\right)^{\beta} \pi^2 \phi(\xi) + \frac{\pi}{\sin(\pi\beta)} \int_{-1}^{1} \phi(y)\, dy$$

$$+ \pi \cot(\pi\beta) \int_{-1}^{1} \frac{\phi(y)\, dy}{y - \xi} \left[(1 - y)\left(\frac{1+y}{1-y}\right)^{\beta} - (1 - \xi)\left(\frac{1+\xi}{1-\xi}\right)^{\beta}\right]$$

$$= \int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{1/2-\beta} \frac{\sqrt{1-x^2}\, f(x)}{x - \xi}\, dx. \tag{10}$$

The relation (10) is satisfied by the solution ϕ of the integral equations (5) for all β such that $0 < \mathrm{Re}(\beta) < 1$.

We have now to choose the unknown constant $\beta\,(0 < \mathrm{Re}(\beta) < 1)$ in such a way that (10) produces an inversion formula for (5). We observe that the only choice is that

$$\beta = \tfrac{1}{2}, \tag{11}$$

and then the function ϕ is recovered from (10) as

$$\phi(\xi) = \frac{A}{\pi\sqrt{1-\xi^2}} - \frac{1}{\pi^2\sqrt{1-\xi^2}} \int_{-1}^{1} \frac{\sqrt{1-x^2}\, f(x)}{x - \xi}\, dx \tag{12}$$

where

$$A = \int_{-1}^{1} \phi(y)\, dy. \tag{13}$$

The reader is referred to the work of Peters[8] for the comparison between the form of the solution (12) of (5) and the solution obtained by Peters for the singular integral equation

$$\int_{0}^{1} \frac{\psi(y)\, dy}{y - \xi} = g(\xi).$$

We observe, from (12) and (13), that if ϕ is an odd function, then the solution of (5) is given by

$$\phi(x) = T^{-1} f - \frac{1}{\pi^2\sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-y^2}\, f(y)}{y - x}\, dy. \tag{14}$$

We next consider eqn. (2).

Setting, as in Williams[10],

$$\eta = (1 - y^2)^{1/2}, \ \sigma = (1 - x^2)^{1/2},$$
$$\phi(y) = (1 - y^2)^{-1/2} \psi[(1 - y^2)^{1/2}], \quad g(x) = h[(1 - x^2)^{1/2}]$$
$$\phi(-y) = -\phi(y), \quad g(-y) = g(y), y > 0 \tag{15}$$

eqn. (2) can be reduced first to the equivalent integral equation

$$a^2(1 - x^2)^{1/2} \int_{-1}^{1} \frac{\varphi(y)\, dy}{y - x} + \int_{-1}^{1} \frac{(1 - y^2)^{1/2} \phi(y)\, dy}{y - x} = \frac{2g(x)}{p + 1}, \tag{16}$$

where

$$a^2 = (1 - p)/(1 + p). \tag{17}$$

We now apply the PBF (1) to (16) in the following manner

Multiply both sides of (16) by

$$\left(\frac{1 - x}{1 + x}\right)^{1/2 - \beta} \frac{dx}{x - \xi},$$

$\xi \epsilon (-1, 1)$ and integrate with respect to x between $x = -1$ to $x = +1$. Here β is a constant ($0 \leqslant \operatorname{Re} \beta < 1$) which will be chosen later on so that a solution of (16) is finally arrived at. We obtain, after utilizing the PBF (1),

$$-(1 - \xi)\left(\frac{1 + \xi}{1 - \xi}\right)^{\beta} (1 + a^2)\pi^2 \phi(\xi)$$
$$+ a^2 \left[\int_{-1}^{1} \frac{\varphi(y)\, dy}{y - \xi} \left\{\int_{-1}^{1} (1 - x)\left(\frac{1 + x}{1 - x}\right)^{\beta} \left(\frac{1}{y - x} + \frac{1}{x - \xi}\right) dx\right\}\right]$$
$$+ \int_{-1}^{1} \frac{(1 - y^2)^{1/2} \phi(y)\, dy}{y - \xi} \left\{\int_{-1}^{1} \left(\frac{1 - x}{1 + x}\right)^{1/2 - \beta} \left(\frac{1}{y - x} + \frac{1}{x - \xi}\right) dx\right\}$$
$$= g_1(\xi), \tag{18}$$

where

$$g_1(\xi) = \frac{2}{p + 1} \int_{-1}^{1} \left(\frac{1 - x}{1 + x}\right)^{1/2 - \beta} \left(\frac{g(x)}{x - \xi}\right) dx. \tag{19}$$

Changing β to $1-\beta$ in (18), we get

$$-(1-\xi)\left(\frac{1+\xi}{1-\xi}\right)^{1-\beta}(1+a^2)\pi^2\phi(\xi)$$

$$+\alpha^2\left[\int_{-1}^{1}\frac{\varphi(y)\,dy}{y-\xi}\left\{\int_{-1}^{1}(1-x)\left(\frac{1+x}{1-x}\right)^{1-\beta}dx\left(\frac{1}{y-x}+\frac{1}{x-\xi}\right)\right\}\right]$$

$$+\int_{-1}^{1}\frac{(1-y^2)^{1/2}\varphi(y)\,dy}{y-\xi}\left\{\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\beta-1/2}\left(\frac{1}{y-x}+\frac{1}{x-\xi}\right)dx\right\}$$

$$=g_2(\xi), \tag{20}$$

where

$$g_2(\xi)=\frac{2}{p+1}\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\beta-1/2}\frac{g(x)}{x-\xi}\,dx. \tag{21}$$

If we now make use of the result (8) to evaluate the inner integrals occurring in the left of (18) and (20), we obtain

$$-(1-\xi)\left(\frac{1+\xi}{1-\xi}\right)^{\beta}(1+a^2)\pi^2\phi(\xi)+\frac{a^2\pi}{\sin(\pi\beta)}\int_{-1}^{1}\phi(y)\,dy$$

$$-a^2\pi\cot(\pi\beta)\left[\int_{-1}^{1}\frac{\varphi(y)\,dy}{y-\xi}\left\{(1-\xi)\left(\frac{1+\xi}{1-\xi}\right)^{\beta}-(1-y)\left(\frac{1+y}{1-y}\right)^{\beta}\right\}\right]$$

$$+\pi\tan(\pi\beta)\int_{-1}^{1}\frac{(1-y^2)^{1/2}\phi(y)\,dy}{y-\xi}\left\{\left(\frac{1+\xi}{1-\xi}\right)^{\beta-1/2}-\left(\frac{1+y}{1-y}\right)^{\beta-1/2}\right\}$$

$$=g_1(\xi) \tag{22}$$

and

$$-(1-\xi)\left(\frac{1+\xi}{1-\xi}\right)^{1-\beta}(1+a^2)\pi^2\phi(\xi)+\frac{a^2\pi}{\sin(\pi\beta)}\int_{-1}^{1}\phi(y)\,dy$$

$$+a^2\pi\cot(\pi\beta)\left[\int_{-1}^{1}\frac{\varphi(y)\,dy}{y-\xi}\left\{(1-\xi)\left(\frac{1+\xi}{1-\xi}\right)^{1-\beta}-(1-y)\left(\frac{1+y}{1-y}\right)^{1-\beta}\right\}\right]$$

$$-\pi\tan(\pi\beta)\int_{-1}^{1}\frac{(1-y^2)^{1/2}\varphi(y)\,dy}{y-\xi}\left[\left(\frac{1+\xi}{1-\xi}\right)^{1/2-\beta}-\left(\frac{1+y}{1-y}\right)^{1/2-\beta}\right]$$

$$=g_2(\xi), \tag{23}$$

respectively.

Multiplying (22) by $\left(\frac{1-\xi}{1+\xi}\right)^{\beta}$ and (23) by $\left(\frac{1-\xi}{1+\xi}\right)^{1-\beta}$ and adding we get,

$$
\begin{aligned}
&-2\pi^2(1+a^2)\,\phi(\xi)(1-\xi) \\
&= \left(\frac{1-\xi}{1+\xi}\right)^{\beta} g_1(\xi) + \left(\frac{1-\xi}{1+\xi}\right)^{1-\beta} g_2(\xi) \\
&- \frac{\pi\, Ca^2}{\sin \pi\beta}\left\{\left(\frac{1-\xi}{1+\xi}\right)^{\beta} + \left(\frac{1-\xi}{1+\xi}\right)^{1-\beta}\right\} \\
&+ a^2\pi\cot(\pi\beta)\left[\int_{-1}^{1}\frac{\varphi(y)\,dy}{y-\xi}(1-y)\left\{\left(\frac{1-\xi}{1+\xi}\,\frac{1+y}{1-y}\right)^{\beta}\right.\right. \\
&\left.\left. - \left(\frac{1-\xi}{1+\xi}\,\frac{1+y}{1-y}\right)^{1-\beta}\right\}\right] - \pi\tan(\pi\beta)\left[\int_{-1}^{1}\frac{\varphi(y)\,dy}{y-\xi}(1-y^2)^{1/2}\right. \\
&\times\left.\left\{\left(\frac{1-\xi}{1+\xi}\right)^{\beta}\left(\frac{1+y}{1-y}\right)^{\beta-1/2} - \left(\frac{1-\xi}{1+\xi}\right)^{1-\beta}\left(\frac{1+y}{1-y}\right)^{1/2-\beta}\right\}\right], \qquad (24)
\end{aligned}
$$

where $C = \int_{-1}^{1} \varphi(y)\,dy$

Now noting that

$$(1-y^2)^{1/2} = (1-y)\left(\frac{1+y}{1-y}\right)^{1/2},$$

we observe from (24), that the solution of the integral equation (16) will be obtained from (24) itself, if we choose β such that

$$a^2\cot(\pi\beta) = \tan\pi\beta$$

i.e., $$\tan\pi\beta = |a| \qquad (25)$$

and, in that event, the solution to (16) is given by

$$
\begin{aligned}
\phi(y) = -\frac{1}{2\pi^2(1+a^2)(1-y)}&\left[\left(\frac{1-y}{1+y}\right)^{\beta} g_1(y) + \left(\frac{1-y}{1+y}\right)^{1-\beta} g_2(y)\right. \\
&\left.- \frac{\pi Ca^2}{\sin\pi\beta}\left\{\left(\frac{1-y}{1+y}\right)^{\beta} + \left(\frac{1-y}{1+y}\right)^{1-\beta}\right\}\right]. \qquad (26)
\end{aligned}
$$

Now, in our case, ϕ is an odd function [cf. eqn. (15)] and, therefore, $C = 0$ and we obtain the solution of (16) as

$$\phi(y) = -\frac{1}{2\pi^2(1+a^2)(1-y)}\left[\left(\frac{1-y}{1+y}\right)^{\beta} g_1(y) + \left(\frac{1-y}{1+y}\right)^{1-\beta} g_2(y)\right] \qquad (27)$$

where the functions g_1 and g_2 are defined as in (19) and (21). We observe that the form (27) of the solution of (16) agrees with that obtained by Lewin.

3. Reduction of equation (16) to a Hilbert problem and its solution

To reduce eqn. (16) to a simple system of Hilbert problems, we shall make use of the inverse operator T^{-1} of the operator T in (4), as given by (14).

We set

$$\chi(x) = \frac{1}{\pi^2 \sqrt{1-x^2}} \int_{-1}^{1} \frac{(1-y^2)^{1/2}\,\phi(y)\,dy}{y-x}. \tag{28}$$

Then, by (4) and (14), we obtain, since ϕ is an odd function,

$$\int_{-1}^{1} \frac{\chi(y)\,dy}{y-x} = -\phi(x). \tag{29}$$

By (28) and (29), we then observe that eqn. (16) is equivalent to the following two coupled integral equations for the two unknown functions $\phi(x)$ and $\chi(x)$

$$\frac{a^2}{\pi^2} \int_{-1}^{1} \frac{\varphi(y)\,dy}{y-x} + (\chi)\,x = \frac{2g(x)}{\pi^2(p+1)\sqrt{1-x^2}} \tag{30}$$

and

$$\int_{-1}^{1} \frac{\chi(y)\,dy}{y-x} + \phi(x) = 0. \tag{31}$$

The two integral eqns. (30) and (31) can finally be reduced to a system of Hilbert problems with constant coefficients by employing the usual method of such reduction as explained in Mikhlin's book[5].

Setting

$$\Phi(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\phi(y)\,dy}{y-z}$$

$$X(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\chi(y)\,dy}{y-z} \tag{32}$$

and employing the Plemelj's formulae for the sectionally analytic functions $\Phi(z)$ and $X(z)$, analytic in the complex z-plane, cut along the real axis between -1 and $+1$, we find that (30) and (31) reduce to the following Hilbert problem

$$a^2 \pi i [\Phi^+(x) + \Phi^-(x)] + \pi^2 [X^+(x) - X^-(x)] = \frac{2g(x)}{(p+1)\sqrt{1-x^2}}$$

and

$$\pi i [X^+(x) + X^-(x)] + [\Phi^+(x) - \Phi^-(x)] = 0 \tag{33}$$

where $F^{\pm}(x)$ are the limiting values of the sectionally analytic function $F(z)$ on the two sides of the cut, as understood in the usual way (see Miklin[5]).

We now decouple the system of Hilbert problems (33) by definining

$$\lambda(z) = (a+i)[a\,\Phi(z) - \pi X(z)]$$

and

$$\mu(z) = (a-i)[a\Phi(z) + \pi X(z)]. \tag{34}$$

Then, in terms of the limiting values λ^+, λ^-, μ^+ and μ^-, the system (33) reduces to the equivalent system,

$$\lambda^+(x) + \frac{a-i}{a+i}\lambda^-(x) = \frac{2g(x)}{\pi i(p+1)(1-x^2)^{1/2}} \tag{35}$$

and

$$\mu^+(x) + \frac{a+i}{a-i}\mu^-(x) = \frac{2g(x)}{\pi i(p+1)(1-x^2)^{1/2}}.$$

We note that, from (34),

$$2a(a^2+1)\,\Phi(z) = (a-i)\lambda(z) + (a+i)\mu(z). \tag{36}$$

Then using the Plemelj formulae in (36) and utilizing (35), the solution of the integral equation (16) can be expressed as,

$$2a(a^2+1)\,\phi(x) = 2[\mu^+(x) + \lambda^+(x)] - \frac{4ag(x)}{\pi i(p+1)(1-x^2)^{1/2}}. \tag{37}$$

Thus, the problem of solving (16) reduces to that of determining the solution of the simplest of the Hilbert problems as given by (35), whose solution can be expressed (see Muskhelishvili[6]) as

$$\lambda(z) = \lambda_0(z)\frac{1}{2\pi i}\int_{-1}^{1}\frac{2g(t)\,dt}{\pi i(p+1)(1-t^2)^{1/2}\lambda_+^0(t)(t-z)} \tag{38}$$

and

$$\mu(z) = \mu_0(z) \frac{1}{2\pi i} \int_{-1}^{1} \frac{2g(t)\, dt}{\pi i (p + 1)(1 - t^2)^{1/2} \mu_+^0(t)(t - z)}$$

where $\lambda_0(z)$ and $\mu_0(z)$ are the solutions, of the homogeneous problem (35), given by

$$\lambda_0(z) = \left(\frac{z - 1}{z + 1}\right)^{\beta}$$

and

$$\mu_0(z) = \left(\frac{z - 1}{z + 1}\right)^{-\beta} \tag{39}$$

where β is given by the relation

$$a = \tan \pi\beta. \tag{40}$$

Noting, then, that

$$\lambda_0^+(x) = \left(\frac{1 - x}{1 + x}\right)^{\beta} e^{\beta\pi i} \tag{41}$$

and

$$\mu_0^+(x) = \left(\frac{1 - x}{1 + x}\right)^{-\beta} e^{-\beta\pi i}$$

the solution (37) of eqn. (16) is given by

$$(a^2 + 1)\phi(x) = -\frac{1}{\pi^2(p + 1)}\left[\lambda_0^+(x) \int_{-1}^{1} \frac{g(t)\, dt}{(1 - t^2)^{1/2} \lambda_0^+(t)(t - x)} + \mu_0^+(x) \int_{-1}^{1} \frac{g(t)\, dt}{(1 - t^2)^{1/2} \mu_0^+(t)(t - x)}\right] \tag{42}$$

or,

$$(a^2 + 1)\phi(x) = -\frac{1}{\pi^2(p + 1)}\left[\left(\frac{1 - x}{1 + x}\right)^{\beta} \int_{-1}^{1} \frac{g(t)}{(1 - t^2)^{1/2}} \left(\frac{1 - t}{1 + t}\right)^{-\beta} \frac{dt}{t - x} + \left(\frac{1 - x}{1 + x}\right)^{-\beta} \int_{-1}^{1} \frac{g(t)}{(1 - t^2)^{1/2}} \left(\frac{1 - t}{1 + t}\right)^{\beta} \frac{dt}{t - x}\right]. \tag{43}$$

Note that the form of the solution (43) is not the same as those obtained in (27) and by Lewin earlier. But this form of the solution does not have the apparent singularity as the form (27) is having at $x = 1$, which was felt and mentioned by Lewin.

However, it is not difficult to obtain the form (27) of the solution of (16) by (42) if we choose appropriate $\lambda_0(z)$ and $\mu_0(z)$ satisfying the homogeneous Hilbert problems (35). We observe that if we choose

$$\lambda_0(z) = \frac{1}{1-z}\left(\frac{z-1}{z+1}\right)^{\beta} = -\frac{1}{1+z}\left(\frac{z-1}{z+1}\right)^{\beta-1}$$

and

$$\mu_0(z) = \frac{1}{1+z}\left(\frac{z-1}{z+1}\right)^{-\beta} = \frac{-1}{1-z}\left(\frac{z-1}{z+1}\right)^{1-\beta} \tag{44}$$

then, by (42), $\phi(x)$ is given by

$$(a^2+1)\,\phi(x) = -\frac{1}{\pi^2(p+1)}\frac{1}{(1-x)}\left[\left(\frac{1-x}{1+x}\right)^{\beta}\int_{-1}^{1} g(t)\left(\frac{1-t}{1+t}\right)^{1/2-\beta}\frac{dt}{t-x}\right.$$

$$\left. + \left(\frac{1-x}{1+x}\right)^{1-\beta}\int_{-1}^{1} g(t)\left(\frac{1-t}{1+t}\right)^{\beta-1/2}\frac{dt}{t-x}\right]; \tag{45}$$

The form (45) is exactly the one as given by (27).

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