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Short Communication

On nonlinear oscillation problem

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Abstract

In the present paper we have given the solution of the differential equation

$$\ddot{x} + f(x) = 0$$

of the general free oscillations where,

$$f(x) = w_0^2\, x^r\, {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix}\, cx^s\right]$$

by applying the linear orthogonal polynomial approximation. The results obtained are of general character and include as particular cases many of the results given earlier by Garde and Saxena and Kushwaha.

Key words : Nonlinear oscillations, orthogonal polynomials, amplitude dependent approximation, generalised hypergeometric function, confluent hypergeometric function.

1. Introduction

In 1959, by the application of Tchebicheff polynomial approximation to $\sin\theta$ in the interval $(-A, A)$, Denmann[1] obtained an amplitude dependent approximation to the frequency of the simple pendulum whose amplitude of motion is A. Later, in 1964, Denmann and Howard[2], Denmann and Liu[3] have applied ultraspherical polynomials to the same problem. Garde[4] in 1965, applied Gegenbauer polynomials to some forced oscillation problem and in 1967, Jacobi polynomials to obtain an approximate solution depending on the amplitude of the nonlinear oscillations defined by the differential equation

$$\ddot{x} + ax + bx^3 = 0,$$

In 1970, Saxena and Kushwaha in two of their joint papers attempted Jacobi polynomials to obtain the amplitude dependent linear approximate solution of the differential equations

$$\ddot{x} + ax + bx^{s} = 0$$

and

$$\ddot{x} + w_0^2 x^r {}_1F_1(\gamma;\ \delta;\ cx^s) = 0.$$

We have attempted in this paper a set of general orthogonal polynomials $\{\phi_n(x)\}$ to give an amplitude dependent linear approximate solution of a general differential equation

$$\ddot{x} + w_0^2 x^r {}_pF_q\left(\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix};\ cx^s\right) = 0$$

where r and s are positive integers and

$${}_pF_q\left[\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix};\ cx^s\right]$$

is the generalised hypergeometric function[7] (p. 73).

The initial conditions of motion are $x = A$, $\dot{x} = 0$ when $t = 0$, A being the amplitude of the motion under which the solution of the proposed problem will be obtained.

The main result of the paper is a generalisation of the results given by Garde[5] and Saxena and Kushwaha[8, 9]. The results obtained are believed to be new.

2. Orthogonal polynomials and linear approximation

Let $\phi_n(x)$ be a polynomial of degree precisely n and $\{\phi_n(x)\}$ forms a set of orthogonal polynomials in the interval (a, b) with respect to the weight function $w(x) > 0$, then

$$\int_a^b w(x)\, \phi_n(x)\, \phi_m(x)\, dx = 0,\ m \neq n$$

$$= K_n\,(\text{say}) \neq 0,\ m = n. \tag{2.0}$$

It can be easily seen that the set of polynomials $\{\phi_n(x/A)\}$ are orthogonal in the interval (aA, bA) with weight function $w(x/A)$.

Let L_w^2 be the class of functions f for which

$$\int_a^b f^2\, w dx < \infty \tag{2.1}$$

and let $\{\phi_n(x)\}$ be an orthonormal system of polynomials which belong to L^2_w. Then the system is closed and for every

$$f \in L^2_w, \ \sum_n a_n^2 = \int_a^b f^2 w dx \quad \text{and} \quad \sum_{n=0}^{\infty} a_n \phi_n \tag{2.2}$$

converges in mean to f

In reference to the known solution of the differential equation

$$\ddot{x} + mx = n. \tag{2.3}$$

We obtain an approximate solution of the problem

$$\ddot{x} + f(x) = 0$$

by truncating the series (2.2) of $f(x)$ after second term. Thus the desired approximation of $f(x)$ in this problem is given by

$$[f(x)]_* = a_0\phi_0\left(\frac{x}{A}\right) + a_1\phi_1\left(\frac{x}{A}\right). \tag{2.4}$$

3. Solution of the main problem

In this section we have solved the differential equation

$$\ddot{x} + f(x) = 0 \tag{3.1}$$

where

$$f(x) = w_0^2 x^r {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} ; cx^s\right]$$

and

$$[f(x)]_* = -K^2 + K^{*2} x,$$

where

$$-K^2 = a_0 + a_1 c, \ K^{*2} = a_1 \frac{d}{A},$$

$$a_j = \frac{A^r w_0^2}{K_j} \int_a^b x^r w(x) {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} ; cx^s\right] \phi_j(x)\, dx,$$

$j = 0, 1$ and $\phi_0 = 1$, $\phi_1 = c + dx$, K_j are defined by the relation (2.0).

The integrals in a_j exist since the series

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} : cx^s\right]$$

is locally uniformly convergent for $p \leqslant q$ if $0 < x \leqslant M$, where M is arbitrary and for $p = q + 1$ if $| cx^s | < 1$. Also, $w(x) > 0$ and each of the polynomials $\phi_0, \phi_1, \ldots$ exist and r is a positive integer. On replacing $f(x)$ by its approximation $[f(x)]_*$ in (3.1), the equation transforms into

$$\ddot{x} + K^{*2} x = K^2$$

which has the solution

$$x_* = \left[A - \frac{K^2}{K^{*2}} \right] \cos K^* t + \frac{K^2}{K^{*2}}$$

under the initial conditions $x = A$, $\dot{x} = 0$ when $t = 0$. This solution is therefore an approximate solution of the problem (3.1) under these conditions.

Obviously it has the approximate period

$$T = \frac{2\pi}{K^*}.$$

Particular cases: (i) If we take $\phi_n(x)$ to be the Jacobi polynomials [ref. 7, p. 254; ref. 10, p. 58], and use the integral [ref. 11, p. 466], and [ref. 10, p. 71], $P_0^{(\alpha,\beta)}(x) = 1$, $P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha + \beta + 2) x + (\alpha - \beta)/2$, then the approximate solution of the nonlinear differential equation

$$\ddot{x} + f(x) = 0$$

where

$$f(x) = w_0^2 x^r \, {}_pF_q \left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} ; cx^s \right]$$

is given by

$$x_* = \left[A - \frac{(\beta - \alpha) A}{(\alpha + \beta + 2)} \left(1 + \frac{\lambda^{*2}}{\lambda^2} \right) \right] \cos \lambda t + \frac{(\beta - \alpha) A}{(\alpha + \beta + 2)} \left(1 + \frac{\lambda^{*2}}{\lambda^2} \right)$$

where

$$\lambda^2 = w_0^2 A^{r-1} \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_j \, c^j A^{sj} (r + sj) \, \Gamma(\beta + r + sj + 1) \, \Gamma(\alpha + \beta + 4)}{\prod_{j=1}^{q} (b_i)_j \, j! \; \Gamma(\beta + 2) \, \Gamma(\beta + \alpha + r + sj + 3)}$$

$$\times \, {}_2F_1 \left(\begin{matrix} 1 - r - sj, \alpha + 2 \\ - \beta - r - sj \end{matrix} ; -1 \right)$$

and

$$\lambda^{*2} = \frac{A^{r-1} w_0^2}{a - \beta} \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_j \, c^j A^{sj} \, \Gamma(a + \beta + 3)\, \Gamma(\beta + r + sj + 1)}{\prod_{i=1}^{q} (b_i)_j \, j! \; \Gamma(\beta + 1)\, \Gamma(a + \beta + r + sj + 2)}$$

$$\cdot\cdot\; {}_2F_1\left(\begin{matrix} -r - sj,\ a + 1 \\ -\beta - r - sj \end{matrix}\ ; -1\right).$$

The series for λ^2 and λ^{*2} are convergent for $p \leqslant q$ and for $p = q + 1$ if $| cA^s | < 1$.

(ii) The approximate solutions of

$$\ddot{x} + w_0^2 x^r \, {}_1F_1 (a ; b ; cx^s) = 0$$

given by Saxena and Kushwaha[9] (p. 295) and that of

$$\ddot{x} + w_0^2 x - w_0^2 cx^3 = 0$$

by Garde[5] (p. 112, 22) are seen to be easy consequences of our main result.

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