J Indian Inst Sci , 65(B), Feb 1984, pp 1-7


# Water pressure on a broken dam during earthquakes

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Received on May 10, 1983, Revised on February 8, 1984

**Abstract**

An exact method based on two-dimensional potential-flow theory has been applied for solving the problem of determination of the hydrodynamic pressure exerted during earthquakes on a dam whose upstream face is broken in the sense that it makes different angles with the horizontal at different heights The pressure coefficient has been evaluated numerically and the results obtained have been compared with those determined experimentally by earlier workers

**Key words:** Dam, pressure, Schwarz-Christoffel transformation, Poisson's formula, pressure coefficient

## 1. Introduction

A detailed study of the problem associated with the determination of hydrodynamic pressure exerted, during earthquakes, on a straight dam with a vertical upstream face by an incompressible fluid was first made by Westergaard[1] In the case of dams with inclined upstream faces of constant and compound slopes, Zangar[2] determined the hydrodynamic pressure experimentally by using an electrical analogue. An exact analytical solution of the problem based on two-dimensional potential-flow theory has been derived by Chwang[3], in the particular case when the inclined upstream face of the dam has a constant slope

In the present paper, we present a further application of the exact method, due to Chwang, based on two-dimensional potential-flow theory to solve the hydrodynamic pressure problem with a more general configuration of a dam whose upstream face is broken at a known height. Our generalisation in essence assumes the fact that the upstream face of the dam makes two different angles with the horizontal, one at the bottom of the reservoir and the other at a known level above the bottom The method of solution requires, as in the work of Chwang, the use of a Schwarz-Christoffel type of transformation and we have solved the problem by using Poisson's integral formula for an analytic function in a half-plane with known boundary values The pressure coefficients have been evaluated numerically from the final solution and the results have been compared with those observed experimentally by Zangar

## 2. Method of solution

As indicated in fig 1, the upstream face of the dam makes angles $\theta_1 = \alpha\pi$ and $\theta_2 = \gamma\pi$ with the horizontal up to and above a known level $h_1$ Rectangular Cartesian coordinates $(x,y,z)$

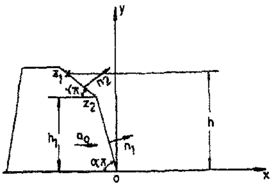


FIG 1. Z-Plane

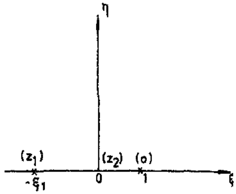


FIG 2 ζ-Plane

are employed throughout. The still water surface occupies the plane $y=h$, the $x$-axis is taken along the bottom of the reservoir and the $z$-axis is assumed to be perpendicular to the $(x,y)$-plane

We assume that the dam undergoes a uniform, horizontal acceleration $a_0$ in the $x$-direction for a short duration of time, and that the resulting flow has all the requirements for the two-dimensional flow theory to be applicable[4]. The mathematical problem then reduces to that of solving Laplace's equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{1}$$

where $p(x,y)$ represents the hydrodynamic pressure at a point $(x,y)$ exerted by the incompressible inviscid fluid in the reservoir. Equation (1) is required to be solved under the following boundary conditions.

$$p\,(x,y) = 0, \,(x \geq h_1\,(\beta_2 - \beta_1) - \beta_2 h,\ y = h) \tag{2a}$$

$$\frac{\partial p}{\partial y} = 0, (x \geq 0, y = 0) \tag{2b}$$

$$\frac{\partial p}{\partial n_1} = -\rho\, a_0 \sin \alpha\pi, (x = -\beta_1 y, 0 \leq y \leq h_1)$$

$$\frac{\partial p}{\partial n_2} = -\rho a_0 \sin \gamma\pi, (x = h_1(\beta_2 - \beta_1) - \beta_2 y, h_1 \leq y \leq h) \tag{2c}$$

where $\bar{n}_1$ and $\bar{n}_2$ denote the inward normal vectors to the two portions of the dam face included between $0 \leq y \leq h_1$ and $h_1 \leq y \leq h$ respectively, with $\beta_1 = \cot\theta_1$ and $\beta_2 = \cot\theta_2$ ($\rho$ denotes the constant density of the fluid).

We construct an analytic function $f(Z)$ of the complex variable $Z = x + iy$ as given by

$$f(Z) = p(x,y) + iq(x,y) \tag{3}$$

where $q(x,y)$ denotes the harmonic conjugate of $p(x,y)$.

The boundary conditions (2b) and (2c) on using Cauchy-Riemann equations give rise to the following conditions for the function $q(x,y)$:

$$\begin{aligned} q(x,y) &= 0, \quad (x \geq 0,\ y = 0) \\ q(x,y) &= -\rho a_0 y \quad (x = -\beta_1 y, \quad 0 \leq y \leq h_1) \\ &= -\rho a_0 (y + h_1)\ (x = h_1 (\beta_2 - \beta_1) - \beta_2 y,\ h_1 \leq y \leq h) \end{aligned} \tag{4}$$

We shall utilize the Schwarz-Christoffel tranformation[5] to transform the given region in the Z-plane on to the upper half-plane of another complex variable $\zeta\,(= \xi + i\eta)$ and shall determine the transformed analytic function $F(\zeta)$, from which the original function $f(Z)$ will be recovered by back transformation. The appropriate transformation to be utilized is given by

$$Z = A \int_{1}^{\zeta} (\zeta + \xi_1)^{\gamma-1} \zeta^{\alpha-\gamma} (\zeta - 1)^{-\alpha}\, d\zeta \tag{5}$$

where the constant $A$ and the real number-$\xi_1\,(\xi_1 > 0)$ are to be determined such that the points -$\xi_1$, 0 and 1 of the $\zeta$-plane are mapped on to the points $Z_1$, $Z_2$ and 0 of the Z-plane (figs 1 and 2) The determination of the constants $A$ and $\xi_1$ is by no means a simple problem and we have shown in the last section of the present paper how these quantities were determined approximately

Using conditions (2a) and (4), the transformed boundary conditions are obtained as

$$Re\,[F(\zeta)] = 0, \quad (-\infty < \xi < -\xi_1, \quad \eta = 0)$$

and

$$Im\,[F(\zeta)] = \begin{cases} -\rho \times a_0\ y(\xi) & (-\xi_1 < \xi < 1, \quad \eta = 0) \\ 0 & (1 < \xi < \infty, \quad \eta = 0) \end{cases} \tag{6}$$

with $y(\xi)$ being determined from the transformation formula (5). If we now use the fact that the analytic function $F(\zeta)$ behaves like $(\zeta + \xi_1)^{1/2}$ at the point -$\xi_1$, we are able to define a new analytic function

$$g(\zeta) = (\zeta + \xi_1)^{-1/2}\ F(\zeta), \tag{7}$$

considering only the positive branch of $(\zeta + \xi_1)^{-1/2}$, and then from equation (7) the boundary conditions (6) take the unified form

$$Im\ g(\zeta) = \begin{cases} -\rho\ a_0\ y(\xi)\ (\zeta + \xi_1)^{-1/2} & (-\xi_1 < \xi < 1,\ \eta = 0) \\ 0 & \text{(otherwise)}. \end{cases} \tag{8}$$

A straightforward application of Poisson's integral formula for an analytic function in a half plane finally determines the function $g(\zeta)$, and hence $F(\zeta)$, in the form.

$$F(\zeta) = \frac{-\rho\, a_0}{\pi} (\zeta+\xi_1)^{1/2} \int_{-\xi_1}^{1} \frac{y(\xi)\, d\xi}{(\xi+\xi_1)^{1/2}\, (\xi-\zeta_1)} \tag{9}$$

The real part of $F(\zeta)$ obtained from equation (9) for $-\xi_1 < \xi < 1$, $\eta = 0$, ultimately gives the expression for $p(\xi)$ as

$$p(\xi) = \frac{A\,\rho a_0}{\pi} \Big[\sin \alpha\pi \int_0^1 (t+\xi_1)^{\gamma-1}\, t^{\alpha-\gamma}\, (1-t)^{-\alpha} \log_e \left| \frac{(t+\xi_1)^{1/2}+(\xi+\xi_1)^{1/2}}{(t+\xi_1)^{1/2}-(\xi+\xi_1)^{1/2}} \right| dt$$

$$+ \sin \gamma\pi \int_0^{\xi_1} (\xi_1 - t)^{\gamma-1}\, t^{\alpha-\gamma}\, (1+t)^{-\alpha} \log_e \left| \frac{(\xi+\xi_1)^{1/2}+(\xi_1-t)^{1/2}}{(\xi+\xi_1)^{1/2}-(\xi_1-t)^{1/2}} \right| dt\Big], \quad (-\xi_1 < \xi < 1) \tag{10}$$

from which we also derive that

$$\frac{dp(\xi)}{d\xi} = \frac{A\,\rho\, a_0}{\pi(\xi+\xi_1)^{1/2}} \Big[\sin \alpha\, \pi \int_0^1 (t+\xi_1)^{\gamma-1/2}\, t^{\alpha-\gamma}\, (1-t)^{-\alpha} \frac{dt}{(t-\xi)}$$

$$- \sin\gamma\pi \int_0^{\xi_1} (\xi_1 - t)^{\gamma-1/2}\, t^{\alpha-\gamma}\, (1+t)^{-\alpha} \frac{dt}{(t+\xi)}\Big] \quad (-\xi_1 < \xi < 1) \tag{11}$$

The singular integrals occurring in equation (11) are next converted into computable non-singular integrals by a standard contour integration procedure[5] and the alternate expressions for $dp(\xi)/d\xi$ in the two cases: (i) $-\xi_1 < \xi \le 0$ and (ii) $0 \le \xi < 1$ are respectively given by

$$\frac{dp(\xi)}{d\xi} = \frac{A\,\rho\, a_0}{\pi(\xi+\xi_1)^{1/2}} \Big[\int_{\xi_1}^{\infty} (t-\xi_1)^{\gamma-1/2}\, t^{\alpha-\gamma}\, (t+1)^{-\alpha} \frac{dt}{(t+\xi)}$$

$$-\pi\cos\gamma\pi\, (\xi_1+\xi)^{\gamma-1/2}\, (-\xi)^{\alpha-\gamma}\, (1-\xi)^{-\alpha}\Big] \quad (-\xi_1 < \xi \le 0) \tag{12a}$$

$$\frac{dp(\xi)}{d\xi} = \frac{A\,\rho\, a_0}{\pi(\xi+\xi_1)^{1/2}} \Big[\int^{\infty} (t-\xi_1)^{\gamma-1/2}\, t^{\alpha-\gamma}\, (t+1)^{-\alpha} \frac{dt}{(t+\xi)}$$

$$-\pi\cos\alpha\pi\, (\xi_1+\xi)^{\gamma-1/2}\, \xi^{\alpha-\gamma}\, (1-\xi)^{-\alpha}\Big] \quad (0 \le \xi < 1) \tag{12b}$$

The expressions (12a) and (12b) are integrated with respect to $\xi$ assuming that $p(\xi)$ is continuous at $\xi = 0$ and we obtain the following computable expressions for $p(\xi)$:

$$p(\xi) = A\,\rho\,a_0\,[2/\pi \int_{\xi_1}^{\infty} (t-\xi_1)^{\gamma-1}\, t^{\alpha-\gamma}\,(t+1)^{-\alpha} \tan^{-1}\frac{(\xi+\xi_1)^{1/2}}{(t-\xi_1)^{1/2}}\,dt$$

$$-\cos\gamma\pi \int_{-\xi_1}^{\xi} (t+\xi_1)^{\gamma-1}(-t)^{\alpha-\gamma}(1-t)^{-\alpha}\,dt] \qquad (-\xi_1 < \xi \leq 0) \qquad (13a)$$

$$p(\xi) = A\,\rho\,a_0\,[2/\pi \int_{\xi_1}^{\infty} (t-\xi_1)^{\gamma-1}\, t^{\alpha-\gamma}\,(t+1)^{-\alpha} \{\tan^{-1}\frac{(\xi+\xi_1)^{1/2}}{(t-\xi_1)^{1/2}}$$

$$-\tan^{-1}\frac{\xi_1^{1/2}}{(t-\xi_1)^{1/2}}\}\,dt - \cos\alpha\pi \int_{0}^{\xi}(t+\xi_1)^{\gamma-1}\, t^{\alpha-\gamma}(1-t)^{-\alpha}\,dt] + p(0),$$

$$(0 \leq \xi < 1) \qquad (13b)$$

The constant $p(0)$ appearing in equation (13b) is the value of $p(\xi)$ at $\xi = 0$, as obtained from equation (13a).

The pressure coefficient defined by

$$p = C_p\,\rho\,a_0 h \qquad (14)$$

can now be computed from the expressions (13a) and (13b) in the range $-\xi_1 < \xi < 1$ and the results so obtained are represented graphically in figs. 3 and 4 where, for the purpose of comparison, we have also shown the graphs drawn by using results experimentally determined by Zangar.

We find that there is fine agreement between the results obtained by the present approach and those obtained by Zangar.

### 3. Determination of $A$ and $\xi_1$

Using the fact that the points $-\xi_1$ and 0 of the $\zeta$- plane are mapped on to the points $Z_1$ and $Z_2$ of the Z-plane we have the relations that

$$\frac{(h-h_1)}{\sin\gamma\pi} = A\int_0^{\xi_1} (\xi_1-\xi)^{\gamma-1}\,\xi^{\alpha-\gamma}\,(\xi+1)^{-\alpha}\,d\xi$$

and (15)

$$h_1/\sin\alpha\pi = A\int_0^{1} (\xi+\xi_1)^{\gamma-1}\,\xi^{\alpha-\gamma}\,(1-\xi)^{-\alpha}\,d\xi$$

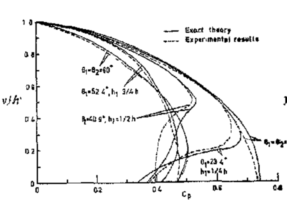


FIG 3 Comparison of the values of pressure coefficient '$C_p$' determined by exact method with those determined experimentally by Zangar.
$\theta_2 = 90°$, $\theta_1$ & $h_1$ variable.

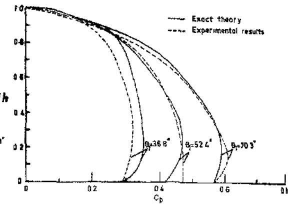


FIG 4 Comparison of the values of pressure coefficient '$C_p$' determined by exact method with those determined experimentally by Zangar
$\theta_2 = 90°$, $h_1 = 3/4\ h$ & $\theta_1$ variable

Also, from equations (15) we obtain the following relation:

$$I_1 - \lambda\, \xi_1^{\alpha} I_2 = 0, \tag{16}$$

where

$$I_1 = \int_0^1 \xi^{\alpha-\gamma} (1-\xi)^{-\alpha} (\xi + \xi_1)^{\gamma-1}\, d\xi$$

$$I_2 = \int_0^1 \xi^{\alpha-\gamma} (1-\xi)^{\gamma-1} (1 + \xi\, \xi_1)^{-\alpha}\, d\xi \tag{17}$$

and

$$\lambda = \frac{h_1 \sin(\gamma\pi)}{(h-h_1)\sin(\alpha\pi)} \tag{18}$$

Expressing the integrals in equations (17) in terms of hypergeometric functions[5] and employing standard transformations[6] to the hypergeometric functions, equation (16) can be written as

$$B(\alpha-\gamma+1, 1-\alpha)\,(1+\xi_1)^{\gamma-1}\, A''\ F\left(1-\gamma, 1-\alpha, 1-\alpha; \frac{\xi_1}{1+\xi_1}\right)$$

$$+\ B(\alpha-\gamma+1, 1-\alpha)\,(1+\xi_1)^{\gamma-\alpha-1}\, \xi_1^{\alpha}\, B''\ F\left(\alpha-\gamma+1, 1, 1+\alpha; \frac{\xi_1}{1+\xi_1}\right)$$

$$-\lambda\, \xi_1^{\alpha} (1+\xi_1)^{-\alpha}\, B(\alpha-\gamma+1, \gamma)\ F\left(\alpha, \gamma, 1+\alpha; \frac{\xi_1}{1+\xi_1}\right) = 0, \tag{19}$$

where

$$A'' = \frac{\Gamma(2-\gamma)\,\Gamma(\alpha)}{\Gamma(\alpha-\gamma+1)} \quad \text{and} \quad B'' = \frac{\Gamma(2-\gamma)\ \Gamma(-\alpha)}{\Gamma(1-\gamma)\,\Gamma(1-\alpha)}$$

$\Gamma(Z)$ and $B(p,q)$ denoting the gamma and the beta functions respectively.

Using the series representations of the hypergeometric functions, equation (19) can be approximated to a polynomial equation in $\xi_1$, which can then be solved numerically. The value of $A$ can next be determined for a known $\xi_1$ through any one of the equations (15). Using this approach, the numerical values of $\xi_1$ and $A$ have been obtained and these are tabulated (Table I) for different sets of values of $\alpha$ and $h_1$ and for $\gamma = 1/2$.

**Table I**

**Numerical values of $\xi_1$ and $A$**

| $\theta_2 = 90°$, $\gamma = 1/2$, $h = 10\ 0$ | | | | |
|---|---|---|---|---|
| $\theta_1$ | $\alpha$ | $h_1$ | $\xi_1$ | $A$ |
| 23 4° | 0 13 | 2 5 | 0 45392 | 3 18319 |
| 40 9° | 0 2272 | 5 0 | 0 14191 | 3 12279 |
| 52 4° | 0 2911 | 7.5 | 0 02481 | 3 18106 |
| 36 8° | 0 2044 | 7 5 | 0 00347 | 3 16291 |
| 70.3° | 0 3906 | 7 5 | 0 07991 | 3 18336 |

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