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Generation of SH type of waves due to stress discontinuity in a poroelastic-layered medium

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Abstract:

The problem of generation of SH waves due to stress discontinuity in a layered homogeneous isotropic poroelastic half-space is solved and the displacement at a point on the free surface is evaluated and presented graphically.

Key words: Poroelasticity, stress discontinuity

1. Introduction

The mechanical behaviour of poroelastic bodies attracted the attention of many scientists and engineers because of its application in soil mechanics, seismology and biomechanics. In the following, the basic equations of Biot's[1] model of a nondissipative poroelastic medium are used.

The study of generation of waves over a half-space dates back to early part of this century and initially the problems studied were of surface normal load sources[2]. Later SH wave generation due to shear stresses was studied by Sezawa[3], Pekeries[4], Nag[5,6] and others. In the following note, we discuss the problem of generation of SH type waves due to a shear stress discontinuity at the interface of a layered half-space of poroelastic materials. This type of shearing stress discontinuity may occur between two layers of the earth having some liquid locked in between them or while cracks propagate during earthquakes.

Following Garvin[7] and Nag[6], the displacement is evaluated at a point of the free surface and presented graphically. Two types of shearing stresses are discussed. It is observed that till a new pulse arrives the displacement goes on increasing and then suddenly falls. Till the discontinuity arrives at the point under consideration from its starting point, the displacement keeps up the above trend. Making the material properties same for both layer and half-space, we obtain the displacement at the free surface corresponding to a sudden introduction of shearing stress discontinuity and expanding or moving uniformly after creation, inside a semi-infinite isotropic poroelastic medium.

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2. Formulation and solution of the problem

Consider a homogeneous isotropic poroelastic layer of thickness h_1 overlying a homogeneous isotropic poroelastic half-space. Let O_{xyz} be a rectangular Cartesian coordinate system such that the xoy plane coincides with the horizontal interface and z-axis is taken into the half-space The discontinuity is to occur suddenly at the interface at the time $t=0$ and moves with a constant velocity U (less than the shear wave velocity of the layer) in the x-direction

Because of the SH motion, v, V are the only non-zero displacements of the solid medium and of the liquid medium respectively. The governing differential equations for the displacements as given by Biot[1] simplify in the present case to

$$N\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) = \frac{\partial^2}{\partial t^2}(\rho_{11} v + \rho_{12} V) \quad 0 = \frac{\partial^2}{\partial t^2}(\rho_{12} v + \rho_{22} V) \tag{2.1}$$

in the absence of dissipation.

Denoting the Laplace transform of $f(t)$ by $f^L(s)$ defined by

$$f^L(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{2.2}$$

so that $f(t) = L^{-1} f^L(s)$, s is real

and again defining complex Fourier transform

$$f^{FL}(\xi, s) = \int_{-\infty}^{\infty} e^{i\xi x} f^L(x,s)\, dx \tag{2.3}$$

On taking Laplace transform with respect to time t, and Fourier transform with respect to the variable x, denoting the transformed variable by v^{FL}, and on using (2.2)-(2.3) on equation (2.1), v_1^{FL} is given by

$$v_1^{FL} = A_1 \cosh(\nu_1 z) + A_2 \sinh(\upsilon_2 z) \quad \text{for the layer} \tag{2.4}$$

and

$$v_2^{FL} = B_1 e^{-\nu_2' z} \quad \text{for the half-space} \tag{2.5}$$

where

$$\upsilon_1^2 = (\xi^2 + s^2/\beta_1^2), \qquad \upsilon_2^2 = (\xi^2 + s^2/\beta_2^2) \tag{2.6}$$

$$\beta_1^2 = \frac{N\rho_{22}}{(\rho_{11}\rho_{22} - \rho_{12}^2)}, \quad \beta_2^2 = \frac{N'\rho_{22}'}{(\rho_{11}'\rho_{22}' - \rho_{12}'^2)} \quad V_1 = -\frac{\rho_{12}}{\rho_{22}} v_1$$

$$\rho_{11}\rho_{22} - \rho_{12}^2 > 0, \ \rho'_{11}\rho'_{22} - \rho'^2_{12} > 0, \ \beta' > \beta \tag{2.7}$$

where N, ρ_{11}, ρ_{22}, ρ_{12} and N', ρ'_{11}, ρ'_{22}, ρ'_{12} are the shear modulus and dynamic mass densities of the layer and half-space, s and ξ are Laplace and Fourier transform parameters, respectively.

The boundary conditions are

(i) $\quad v_1 = v_2$ at $z = 0$ (2.8)

(ii) $\quad \sigma_{yz} = 0$ at $z = -h_1$ (2.9)

(iii) $\quad (\sigma_{yz})_1 - (\sigma_{yz})_2 = L(x,t)\ H(t)$ at $z = 0$ (2.10)

where $H(t)$ is the unit step funtion, $L(x,t)$ is defined in each case separately and the suffix '1' refers to the layer while '2' refers to the half-space.

From equations (2.8)-(2.9) and (2.4)-(2.5) we get,

$$A_1 = A_2 \tag{2.11}$$

$$A_2 \cos(v_1 h_1) + A_1 \sinh(v_2 h_1) \tag{2.12}$$

Two different forms of $L(x,t)$ are considered below:

Case I

Let

$$L(x,t) = \begin{cases} P & a \leq x \leq b + Ut \\ 0 & \text{elsewhere} \end{cases} \tag{2.13}$$

where P, a and b are constants.

From the boundary condition (iii) one gets with the help of (2.13)

$$A_2 N v_1 + B_1 N' v_2 = \frac{P}{2s\pi}\left[\frac{e^{-i\xi a} - e^{-i\xi b}}{i\xi} + \frac{e^{-i\xi b}}{i\xi + s/U}\right] \tag{2.14}$$

Solving equations (2.11), (2.12) and (2.14), A_1, A_2, B_1 are evaluated and on using them, the displacement at free surface is

$$v_1^{FL}(\xi, -h_1, s) = \frac{P}{2\pi s\,[\,N' v_2 \cosh(v_1 h_1) + n\, v_1 \sinh(v_1 h_1)\,]} \times$$

$$\left[\frac{e^{-i\xi a} - e^{-i\xi b}}{i\xi} + \frac{e^{-i\xi b}}{i\xi + s/U} \right]$$

$$= \frac{P}{\pi s} \frac{(1 - K e^{-2v_1 h_1})^{-1}}{(N v_1 + N' v_2)} \left[\frac{e^{-i\xi a} - e^{-i\xi b}}{i\xi} + \frac{e^{-i\xi b}}{i\xi + s/U} \right] \tag{2.15}$$

where

$$K = \left(\frac{N v_1 - N' v_2}{N v_1 + N' v_2} \right), \quad (K < 1) \tag{2.16}$$

Taking the inverse Fourier transformation, we obtain

$$v_1^{FL}(x, -h_1, s) = I_1^L + I_2^L + I_3^L \tag{2.17}$$

where

$$I_1^L(x_1, -h_1, s) = \frac{P}{\pi s} \int_{-\infty}^{\infty} \frac{e^{-v_1 h_1 + \xi x_1}}{i\xi (N v_1 + N' v_2)} \quad [1 + K e^{-2v_1 h_1} + K^2 e^{-4v_1 h_1} + \ldots] \; d\xi$$

$$I_2^L(x_2, -h_1, s) = -\frac{P}{\pi s} \int_{-\infty}^{\infty} \frac{e^{-v_1 h + i\xi x_2}}{i\,(N v_1 + N' v_2)} \quad [1 + K e^{-2v_1 h_1} + K^2 e^{-4v_1 h_1} + \ldots] \; d\xi$$

$$I_3^L(x^2, -h_1, s) = \frac{P}{\pi s} \int_{-\infty}^{\infty} \frac{e^{-v_1 h_1 + i\xi x_2}}{(N v_1 + N' v_2)\,(i\xi + s/u)} \times$$

$$[1 + K e^{-2v_1 h_1} + K^2 e^{-4v_1 h_1} + \ldots] \; d\xi \tag{2.18}$$

where $x_1 = x - a$, $x_2 = x - b$.

Following Garvin[7] and Nag[6] in using the Cagniard's method, the displacement v_1 at the free surface is

$$v_1(x_1, -h_1, t) = \frac{2P\beta_1}{N\pi} \sum_{n=1,3,5,} L^{-1}\,(I_{1,n}^L + I_{2,n}^L + I_{3,n}^L) \tag{2.19}$$

where

$$L^{-1} I_{1,n}(x_1, -h_1, s) = \int_0^t (t-\lambda) F_{1,n}[\alpha_{1,n}(\lambda)]\, \mathrm{d}\lambda, \tag{2.20}$$

$$F_{1,n}[\alpha_{1,n}(t)] = Im\ \{ [(1+\alpha_{1,n}^2)^{1/2} + \iota_3 (\iota_1^2 + \alpha_{1,n}^2)^{1/2}]^{-1} \times$$
$$\alpha_{1,n} K^{n-1/2} \frac{d\,\alpha_{1,n}}{\mathrm{d}t} \} H[t - \frac{(x_1^2 + n^2 h_1^2)^{1/2}}{\beta_1}] \tag{2.21}$$

$$\alpha_{1,n}(t) = \frac{\beta_1}{(x_1^2 + n^2 h_1^2)} \{ i x_1 t + n h_1 \left[t^2 - \frac{(x_1^2 + n^2 h_1^2)}{\beta_1^2} \right]^{1/2} \} \tag{2.22}$$

$$\iota_1 = \beta_1/\beta_2, \quad \iota_2 = \beta_1/U, \quad \iota_3 = N'/N$$

$I_2(x_2, -h_1, t)$, $F_{2,n}\left[\alpha_{2,n}(t)\right]$ are obtained from I_1, $F_{1,n}\left[\alpha_{1,n}\right]$ by replacing x_1 by x_2,

$$L^{-1} I_{3,n}(x_2, -h_1, t) = \int_0^\infty (t-\lambda) F_{3,n}[\alpha_{2,n}(\lambda)]\, \mathrm{d}\lambda \tag{2.23}$$

$$F_{3,n}[\alpha_{2,n}(t)] = Re\ \{ [(1+\alpha_{2,n}^2)^{1/2} + \iota_3 (\iota_1^2 + \alpha_{2,n})^{1/2}]^{-1} \times$$
$$(\iota_2 + i\,\alpha_{2,n})^{-1}\ K^{n-1/2} \frac{\mathrm{d}\alpha_{2,n}}{\mathrm{d}t} \Big\}\ H\left[t - (x_2^2 + n^2 h_1^2)^{1/2} / \beta_1 \right] \tag{2.24}$$

and *Re* and *Im* stand for the real and the imaginary parts of the expressions before which they come.

Case 2

Let

$$L(x,t) = Ph\ S(x - Ut) \quad \text{at } z = 0 \tag{2.25}$$

where P is a constant and $S(x - Ut)$ is Dirac delta function of argument $(x - Ut)$ In the above, the stress discontinuity is created at $x = 0$ and is travelling with uniform velocity U along the x-axis.

Proceeding as in the above case, the non-vanishing displacement in this case is

$$V_1(x, -h_1, t) = 2Ph_1\, \iota_2/\pi N \sum_{n=1,3,5,\ldots} \int_0^t F_n[\alpha_n(\lambda)]\, \mathrm{d}\lambda \tag{2.26}$$

where

$$F_n[\alpha_n(t)] = Re\left\{\left[(1+\alpha_n^2)^{1/2} + \iota_3(\iota_1^2+\alpha_n^2)^{1/2}\right]^{-1}(\iota_2 + \iota\,\alpha_n)^{-1} \times K^{n-1/2}\frac{d\alpha n}{dt}\right\}$$

$$H\left[t - (x^2+n^2h_1^2)^{1/2}/\beta_1\right] \tag{2.27}$$

and

$$\alpha_n(t) = \frac{\beta_1}{(x^2+n^2h_1^2)}\left[\,ixt + nh_1\left\{t^2 - \frac{(x^2+n^2h_1^2)}{\beta_1^2}\right\}^{1/2}\right] \tag{2.28}$$

3. Particular cases

(A) Half-space

When the layer and half-space are having same material, *i.e.*

$$v_1 = v_2,\ N' = N,\ \iota_1 = \iota_3 = 1,\ \iota_2 = \beta_1/U$$

In this case the displacements at the free surface due to sudden introduction of shearing stress discontinuity which expands or moves uniformly after creation, inside a semi-infinite isotropic poroelastic medium have been obtained as particular case of Nag[5]

By taking $a = b$, from equation (2.19) we get

$$v_1(x, -h_1, t) = \frac{P\beta_1}{N\pi}\, L^{-1} I_3^L \tag{3.1}$$

$$I_3^L(x, -h_1, s) = \int_{h_1/\beta_1}^{\infty}\frac{F_1(\alpha)}{s^2}\;\frac{d\alpha}{dt}\; e^{-st}\,dt \tag{3.2}$$

where

$$F_1[\alpha(t)] = Re\,(1+\alpha^2)^{-1/2}(\iota_2 + i\alpha)^{-1} \tag{3.3}$$

where $\alpha(t)$ is given in (2.22).

Now

$\int_{h_1/\beta_1}^{\infty} F_1(\alpha)\,\frac{d\alpha}{dt}\, e^{-st}\,dt$ is the Laplace transform of $G(\alpha)$

where

$$G(\alpha) = \begin{cases} 0 \quad \text{for } 0 < t < h_1/\beta_1 \\ Re\,\{[F_1(\alpha)]\,\dfrac{d\alpha}{dt}\,\} \text{ for } t > h_1/\beta_1 \end{cases} \tag{3.4}$$

Since $F_1(\alpha)$ is real and $d\alpha/dt$ is imaginary or $h_1/\beta_1 < t < (x_1^2 + h_1^2)^{1/2}/\beta_1$, one can replace h_1/β_1 by $[(x_1^2 + h_1^2)^{1/2}/\beta_1]$.

Thus we get $G(\alpha)$,

$$G(\alpha) = \frac{U(x_1^2 + h^2)\{h_1^2 + x_1^2 - U^{tx}{}_1\}\,H[t - (x_1^2 + h_1^2)^{1/2}/\beta_1]}{\beta_1\{t^2 - (x_1^2 + h_1^2)/\beta_1^2\}^{1/2}\,[(h_1^2 + x_1^2 - Utx_1)^2 + U^2h_1^2[t^2 - (x_1^2 + h_1^2)/\beta_1^2]]} \tag{3.5}$$

Taking the inverse Laplace transform for the equation (3 2) we get

$$v_1(x_1, -h_1, t) = \frac{P\beta_1}{N\pi}\int_{(x_1^2 + h_1^2)^{1/2}\beta_1}^{t} (t - \lambda)\, G[\alpha(\lambda)]\, d\lambda \tag{3.6}$$

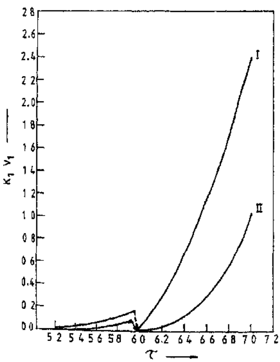

FIG 1. Variation of displacement as a function of time

where $G(\alpha)$ is given in (3.5)

Equation (3 6) coincides with the results given by Nag[5] for case 1.

(B) Layered half-space of elastic medium

If we make $\rho_{11} = \rho_{22}, \rho_{12} = 0$, which implies $\beta_1^2 \to N/\rho_{11}, \beta_2^2 \to N'/\rho'_{11}$ which corresponds with that for the classical case.

4. Discussion

$v_1(x, -h_1, t)$ is evaluated when assuming $\tau = t\beta_1/h_1$, $\iota_1 = 0.54047$, $\iota_2 = \sqrt{2}$, $\iota_3 = 2.93262$, $a = b = 0$ In fig 1, curve I is of case (1) with $K_1 = N\pi/2P\beta_1$ and curve II is of case (2) with $K_1 = N\pi/2\pi h_1^2$. It is observed that till a new pulse arrives the displacement goes on increasing and then suddenly falls. Till the discontinuity arrives at the point under consideration from its starting point the displacement keeps up the above trend. The numerical values are considered from Fatt[8] for the layer and Nowinski[9] for the half-space.

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References

1 BIOT, M A — Theory of propagation of elastic waves in fluid saturated prorous solid, *J Acoust Soc Am*, 1956, **28**, 168

2. EWING, W.M. *et al* — *Elastic waves in layered medium*, McGraw-Hill Book Company, Inc, New York, 1957

3. SEZAWA, K. — Love waves generated from a source of a certain depth, *Bull Earth Res Inst* (Tokyo), 1935, **13**, 1-17

4 PEKERIES, C.L — The propagation of an SH pulse in a layered medium, *Trans Am Geophys Un*, 1941, 392-393

5. NAG, K.R. — Disturbance due to shearing stress discontinuity in a semi-infinite medium, *Geophys J. Royal Astr Soc* 1962, **6**, 468-478.

6. NAG, K.R. — Generation of SH type of waves due to stress discontinuity in a layered medium, *QJMAM*, 1963, **16**(3), 295

7. GARVIN, W W. — Exact transian solution of the burried line source problem, *Proc Royal Soc Am.*, 1966, **234A**, 528.

8. FATT, I. — The Biot-Willis coefficients for a sand stone, *J Appl, Mech*, 1959, **26**, 296

9. NOWINSKI, J L AND DAVIS, C.F. — A model of human skull as a poroelastic spherical shell subjected to a quasi-static load, *Math, Biosci*, 1970, **8**(3/4), 397-416