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Short Communication

An alternative definition of local connectedness in bitopological spaces

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Abstract

In this note we give an alternative definition of local connectedness in a bitopological space and derive some properties of a locally connected bitopological space.

Key words: Bitopological space, connectedness, component, local connectedness

1. Introduction

The notion of connectedness in a bitopological space is due to Pervin[1] The concept of local connectedness in a bitopological space has been introduced by Dasgupta and Lahiri[2] They have also derived some of the basic properties of such a space. However, according to their definition, in a locally connected bitopological space, the two topologies involved coincide and hence we ultimately deal only with local connectedness in a topological space. Therefore, in this note we have made an attempt to give an alternative definition of local connectedness in bitopological space. For a set of basic definitions and notations we refer the reader to Kelly[3] and Pervin[1,4].

2. Alternative definition of local connectedness

In this section, we give an alternative definition of local connectedness in a bitopological space and derive some of the properties of a locally connected space

DEFINITION 2 1 Let (X, P, Q) be a bitopological space. If U is P-open and V is Q-open, then $U \cap V$ is said to be *partially open* in (X, P, Q).

DEFINITION 2.2· A bitopological space (X, P, Q) is called *locally connected at a point* $x \in X$ if and only if for every pair of P-open set U and Q-open set V each containing x, there

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exist a P-open set C and Q-open set D such that $C \cap D$ is connected and $x \in C \cap D \subsetneq U \cap V$ (X, P, Q) is called *locally connected* if and only if it is locally connected at each point of X

Remarks 2 1

(i) Note that when $P = Q$, by taking $V = U$ and $D = C$, the above definition reduces to the definition of local connectedness at a point x in a topological space (X, P)

(ii) Note that we have not assumed that $C \subsetneq U$ and $D \subsetneq V$

(iii) Local connectedness for a bitopological space is not equivalent to the local connectedness of the individual topologies as shown by the following examples

Example 2 1 Let (X, P, Q) be a bitopological space, where

$$X = \{ 1,2,3, \ldots, n, \ldots \},$$

$$P = \{ \phi, X, \{1,3,4, \ldots\}, \{1,4,5, \ldots\}, \ldots, \{1, n, n+1, \ldots\}, \ldots \} \text{ and}$$

$$Q^* = \{ \phi, X, \{2\}, \{3\}, \ldots \{n\}, \ldots \} \text{ is a subbase for } Q$$

Note that the topological space (X, P) and (X, Q) are locally connected whereas the space (X, P, Q) is not locally connected at 1

Example 2 2 Let

$$X = \{ (0,y) \mid -1 \leq y \leq 1 \} \cup \{ (x, \sin 1/x) \mid x > 0 \}$$

Let P be the topology on X, obtained by the restriction of the usual topology on R^2 Let $Q = \{ \phi, X \}$ be the indiscrete topology on X. Then we note that the bitopological space (X, P, Q) is locally connected whereas the toplogical space (X, P) is not locally connected

3. Basic properties

In this section, we derive some basic properties of a locally-connected bitopological space

Theorem 3 1 *A bitopological space* (X,P,Q) *is locally connected if and only if the components of partially open sets can be expressed as a union of connected partially open sets*

Proof We first assume that the bitopological space (X, P, Q) is locally connected Let K be a component of a nonempty partially open set $U \cap V$ If $x \in K$, then there exist a connected partially open set $C_x \cap D_x$ such that $x \in C_x \cap D_x \subsetneq U \cap V$ Since K is a component of $U \cap V$, $x \in C_x \cap D_x \subseteq K$ Varying x over K we get $K = \bigcup_{x \in K} (C_x \cap D_x)$

For the converse, let $x \in U \cap V$, a partially open set Then there is component K of $U \cap V$ such that $x \in K \subsetneq U \cap V$ By assumption, there is a connected partially open set $C \cap D$ such that $x \in C \cap D \subsetneq K$ This proves that the bitopological space (X,P,Q) is locally connected

Theorem 3 2 Let (X,P,Q) *and* (X^*, P^*, Q^*) *be two bitopological spaces and f* (X,P,Q) → (X^*, P^*, Q^*) *be a continuous surjective mapping Further, let* f *be partially open, that is,* f *carries partially open sets in* X *to partially open sets in* X^* *If* (X,P,Q) *is locally connected, then* (X^*, P^*, Q^*) *is also locally connected*

Proof. Let $y \in X^*$ and let $U \cap V$ be a partially open set containing y in (X^*, P^*, Q^*) Since f is continuous, $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ is a partially open set in X Since f is onto, there exists an $x \in X$ such that $f(x) = y$ Since the bitopological space (X, P, Q) is locally connected and $f^{-1}(U) \cap f^{-1}(V)$ is a partially open set containing x, there exists a connected partially open set $C \cap D$ in X such that $x \in C \cap D \subsetneq f^{-1}(u) \cap f^{-1}(v)$ Since f is continuous, $f(C \cap D)$ is connected in (X^*, P^*, Q^*) Also, since f is partially open, $f(C \cap D)$ is partially open in (X^*, P^*, Q^*) The result follows from the fact that

$$y = f(x) \in f(C \cap D) \subsetneq U \cap V$$

We omit the proof of the following theorem, which is straightforward

Theorem 3 3 Let (X_i, P_i, Q_i) *be a countable collection of bitopological spaces and let* (X,P,Q) *be the product space, where* (X,P) *is the product of the collection* (X_i, P_i) *and* (X,Q) *is the product of the collection* (X_i Q_i). *Further, assume that each projection is partially open Then the bitopological space* (X,P,Q) *is locally connected if and only if* (X_i, P_i, Q_i) *is connected for all except a finite number of* i *and the bitopological space* (X_i, P_i, Q_i) *is locally connected for each* i.

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