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A comparison between the 'local potential method' and the 'exact numerical procedure' (Application to the stability analysis of a hot layer of fluid)

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Abstract

In this paper the 'local potential method' as introduced by Prigogine and Glansdorff and applied to hydrodynamic stability problems by Schechter and Himmelblau and the 'exact numerical procedure' used by Harris and Reid and others have been applied to determine the stability criteria of a hot layer of fluid with mean temperature changing at a constant rate, which had already been determined by Krishnamurthi. While solving by the two methods and comparing the results with the known results it is found that unless sufficient representation for the eigenfunctions is considered the 'local potential method' does not give good approximation and unless the initial guess is quite close to the actual result the 'exact numerical procedure' involves long processes. It is found that two methods can be suitably combined to give a useful procedure.

Key words: Local potential method, exact numerical procedure, principle of exchange of stabilities.

1. Introduction

The concept of local potential had been introduced by Prigogine and Glansdorff[1-3] and was applied to hydrodynamic stability problems by Schechter and Himmelblau[4] and Platten[7-8]. The method gives approximate solution by constructing the local potential. The exact numerical procedure involves the transformation of the two-point boundary value problems into an initial value problem and was used by Harris and Reid[5], Chock and Schechter[9] and Sastry and Rao[10]. Vanderborck and Platten[11] had investigated the usefulness of the two methods and in this paper we have investigated considering the case: the hydrodynamic stability of a horizontal layer of fluid with mean temperature changing at a constant rate. The problem had been solved by Krishnamurthi[6] and thus the comparison of the results by the two methods with the well-known results helps us in arriving at some conclusions presented in the last section.

2. Basic equations and boundary conditions

A coordinate system $ox_1x_2x_3$ is chosen having ox_3 vertically upwards and ox_1, ox_2 in the horizontal plane. The fluid is assumed to be infinite in horizontal extent and confined between two rigid, perfectly conducting boundaries $x_3 = 0$ and $x_3 = d$. The lower and upper plates are maintained at constant difference of temperatures $\Delta T > 0$. It is assumed that the temperature in the fluid is changing at a constant rate. The basic equations, with Boussinesq approximations are[6,12]:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial}{\partial x_i} \left(\frac{P}{\rho_0}\right) - g \frac{\rho}{\rho_0} \vec{K} + \gamma \nabla^2 u_i \tag{1}$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = K_t \nabla^2 T \tag{2}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{3}$$

$$\rho = \rho_0 \left[1 - \alpha (T - T_0) \right] \tag{4}$$

where u_i $(i = 1,2,3)$, ν, ρ p, g, K_t and α are the components of the velocity, kinematic viscocity, the density, the pressure, the acceleration due to gravity, thermal diffusivity and the coefficient of volume expansion respectively. $\vec{K}$ being the unit vector in the direction of ox_3 axis.

The boundary conditions: Since the boundaries are rigid and perfectly thermally conducting we have

$$u_i = 0 \quad \text{for} \quad x_3 = 0,d \tag{5}$$

and

$$T = T_0 \text{ for } x_3 = 0 \text{ and } \quad T = T_0 - \Delta T \textit{ for } x_3 = d \quad \text{at time } t = 0 \tag{6}$$

3. Linearized equations and normal mode analysis

In the stationary state, when $u_i = 0$ (i = 1,2,3) and $\bar{T}(x_3,t)$, the mean temperature is changing at a uniform rate q_n; we find from (2):

$$\bar{T}(x_3,t) = Q\eta t + \frac{Q_\eta}{2K_t} x_3^2 - \left(\frac{\Delta T}{d} + \frac{Q_\eta d}{2K_t}\right) x_3 + T_0 \tag{7}$$

Assuming p and ρ to be the values of pressure and density in the undisturbed state and assuming $u_i, p + \delta p, \rho + \delta\rho$ and $\bar{T}(x_3, t) + \theta$ to be the velocity components, pressure, density and temperature in the disturbed state where u_i, δp, $\delta\rho$ and θ are small in magnitudes, we have after linearization, introduction of non-dimensional variables denoted by dashes

$$x_i = dx_i' \ (i = 1,2,3), \quad u_i = \frac{K_t}{d} u_i' \ (i = 1,2,3)$$

$$t = \frac{d^2}{\gamma} t', \quad \theta = (\Delta T)\theta', \quad \delta p = \frac{\rho_0 K_t \gamma}{d^2} \delta p', \quad Q_\eta = \frac{K_t \Delta T}{d^2} Q_\eta'$$

following equations:

$$\frac{\partial u_i}{\partial t} = - \frac{\partial (\delta p)}{\partial x_i} + R\theta\bar{K} + \nabla^2 u_i \tag{8}$$

$$Pr \frac{\partial \theta}{\partial t} = \nabla^2\theta + u_3 + \frac{Q_\eta}{2} (1-2x_3) u_3 \tag{9}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{10}$$

(after dropping the dashes). Here $R = \dfrac{g\alpha \Delta T d^3}{\gamma K_t}$ is the Rayleigh number and $Pr = \gamma/K_t$ is the Prandtl number.

Supposing

$$(u_3, u_1, u_2, \delta p, \theta) = [\ W(x_3), \frac{iK}{a^2} DW(x_3), \frac{im}{a^2} DW(x_3), P(x_3), \Theta(x_3)\ ]$$
$$\exp\ (iKx_1 + imx_2) \tag{11}$$

when the principle of exchange of stabilities is valid and substituting (11) into (8) and (9), we have, after eliminating $P(x_3)$,

$$(D^2 - a^2)^2 W = Ra^2 \Theta \tag{12}$$

$$(D^2 - a^2) \Theta = -W - Q\eta/2 (1 - 2x_3) W \tag{13}$$

and the boundary conditions from (5) and (6), as

$$W = DW = \Theta = 0 \text{ at } x_3 = 0,1 \tag{14}$$

where $D \equiv \dfrac{d}{dx_3}$

4. Solution

4.1 Local potential method

Considering X^0 (which represents W^0 or $\Theta^{(0)}$, the actual solution) and $X = X^0 + \delta X$, a variation satisfying the same boundary conditions, we note that

$$-1/2\partial/\partial t \int_0^1 (\delta X)^2 \, dx_3 = \int_0^1 \frac{dx}{dt}\delta X \, dx_3 - \int_0^1 \frac{dx^0}{dt} \delta X \, dx_3.$$

Taking $X^{(0)}$ to represent $W^{(0)}$ and $\Theta^{(0)}$ in turn, using the above relationship and the equations (12)-(14), we construct Φ_L, the local potential for the present problem, as

$$\begin{aligned}\Phi_L = < a^4 W^{(0)} W - Ra^2 \Theta^{(0)} W - a^2 W^{(0)} \Theta - 1/2 (D^2 W)^2 + 2D^2 W^{(0)} D^2 W \\ + a^2/2 (DW)^2 + a^2 D W^{(0)} DW + a^2/2 (D\Theta)^2 + a^4 \Theta^{(0)} \Theta \\ - a^2 Q_\tau/2 (1 - 2x_3) W^{(0)} \Theta >\end{aligned}$$

where $< >$ represents integration with respect to x_3 from 0 to 1.

We take

$$W = \sum_{i=1}^{N} a_i f_i, \quad \Theta = \sum_{i=1}^{N} b_i t_i \tag{16}$$

$$W^{(0)} = \sum_{i=1}^{N} a_i^{(0)} f_i, \quad \Theta^{(0)} = \sum_{i=1}^{N} b_i^{(0)} t_i \tag{17}$$

where f_i's and t_i's are the sets of trial functions, satisfying the boundary conditions (14). Substituting (16) and (17) into (15), minimizing with respect to a_i's and b_i's finally putting $a_i = a_i^{(0)}$ and $bi = bi^{(0)}$ ($i = 1,2, ..., N$) and later eliminating $a_i^{(0)}$'s and $b_i^{(0)}$'s, we obtain the following ($2N \times 2N$) determinant to vanish

$$\text{Det} \begin{vmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{vmatrix} = 0 \tag{18}$$

where

$$\begin{aligned} A_{ij} &= a^4 < f_i, f_j > + 2a^2 < f_i' f_j' > + < f_i'' f_j'' > \\ B_{ij} &= - a^2 R < f_i t_j > \\ C_{ij} &= - < f_j t_i > - Q_\tau/2 < (1 - 2x_3) f_j t_i > \\ D_{ij} &= < t_i' t_j' > + a^2 < t_i t_j > . \end{aligned} \tag{19}$$

Here the ′ and ″ denote first and second derivatives with respect to x_3. (For details of the method, see refs. 1-4, 7-8).

Since equations (12)—(14) do not show the symmetry properties we take trail functions without such properties, as

$$f_I = x_3^2 (1-x_3)^2 (1-2x_3)^{I-1}$$
$$t_I = x_3 (1-x_3) (1-2x_3)^{I-1} \tag{20}$$

Substitution of (20) into (19), making use of (18) gives R for various values of a, from which R_c, the critical Rayleigh number, and the corresponding value a_c are obtained. The values of R_c and a_c for various values of Q_η are presented in Table I.

4.2. Exact numerical procedure

Following Harris and Reid[5] and Chock and Schechter[9], we convert the boundary value problem (12)—(14) into an initial value problem. We set

$$DY_1 = Y_2,\ DY_2 = Y_3,\ DY_3 = Y_4$$
$$DY_4 = (-a^2 Y_1 + 2Y_3 + RY_5)a^2,\ DY_5 = Y_6$$
$$DY_6 = a^2 Y_5 - Y_1 - Q_\eta/2\,(1-2x_3)Y_1 \tag{21}$$

with

$$Y_1(0) = Y_2(0) = Y_5(0) = 0 \tag{22a}$$

$$Y_1(1) = Y_2(1) = Y_5(1) = 0 \tag{22b}$$

where

$$W = Y_1,\ DW = Y_2,\ D^2W = Y_3,\ D^3W = Y_4$$
$$\Theta = Y_5,\ D\,\Theta = Y_6 \tag{23}$$

For any particular i, Y_i has six linearly independent solutions which we call as $Y_i(j)$ $(j = 1,2, ..., 6)$ and thus the general solution for Y_i is written as

$$Y_i = \sum_{j=1}^{6} C_j Y_i^{(j)} \qquad (i = 1,2,..., 6) \tag{24}$$

We choose

$$Y_i^{(j)}(0) = \delta_{ij} \tag{25}$$

so that $C_1 = C_2 = C_5 = 0$

and hence

$$Y_i = C_3\ Y_i^{(3)} + C_4\ Y_i^{(4)} + C_6\ Y_i^{(6)} \tag{26}$$

and finally from (22b), we get

$$\text{Det} \begin{vmatrix} Y_1^{(3)}(1) & Y_1^{(4)}(1) & Y_1^{(6)}(1) \\ Y_2^{(3)}(1) & Y_2^{(4)}(1) & Y_2^{(6)}(1) \\ Y_5^{(3)}(1) & Y_5^{(4)}(1) & Y_5^{(6)}(1) \end{vmatrix} = 0 \tag{27}$$

Numerical integration of (21) and use of (25) lead to the determination of the elements of the above determinant. Starting with initial guess for R, we use Newton-Raphson method for better approximate values. From the values of R for a's, the critical Rayleigh number R_c and the corresponding value a_c are determined. Values of (R_c, a_c) for various values of Q_η are presented in Table I.

5. Discussion

Proceeding in the same way as Watson[13], it can be proved easily that in this problem positive and negative rate of change of mean temperature have the same effect on the stability criteria. From Table I we find that both the methods considered in this paper show that R_c

Table I

Critical Rayleigh numbers and related constants

(A comparative study of the results)

Krishnamurthi[6]			Present study					
			Local potential method				Exact numerical procedure	
			$N = 2$		$N = 10$			
Q_η	a_c	R_c	a_c	R_c	a_c	R_c	a_c	R_c
0.0	3.12	1708.0	3.12	1749.97	3.12	1707.76	3.117	1707.7677
0.667	3.12	1706.5	3.12	1748.56	3.12	1706.32	3.118	1706.3256
1.20	3.12	1703.3	3.12	1745.41	3.12	1703.12	3.121	1703.1160
2.00	3.12	1695.1	3.12	1737.43	3.13	1694.95	3.126	1694.9565
3.33	3.13	1673.4	3.14	1715.95	3.14	1673.06	3.144	1673.0693
6.00	3.20	1604.3	3.20	1647.35	3.20	1603.43	3.199	1603.4398
8.00	3.24	1537.5	3.25	1580.76	3.25	1536.20	3.249	1536.2356
11.33	3.34	1414.9	3.34	1457.83	3.34	1413.01	3.339	1413.0172
18.00	3.49	1182.0	3.49	1222.01	3.49	1179.31	3.494	1179.3200
38.00	3.70	747.1	3.72	776.82	3.72	744.79	3.724	744.8015
78.00	3.86	419.1	3.84	437.40	3.86	417.62	3.855	417.6287

decreases with increase in Q_η, which is in agreement with the results obtained by Krishnamurthi[6]. Comparing the results obtained by local potential method and the exact numerical procedure, we find that for $N = 2$, the local potential method gives poor approximation whereas for $N = 10$, when the computational work becomes very lengthy it gives sufficiently accurate result. The exact numerical procedure on the other hand gives an accurate result but the process may be quite long unless the initial guess is sufficiently close to the actual values. We find that if the result obtained by local potential method for $N = 2$ (say) is taken as the initial guess for exact numerical procedure, the iterative process can be shortened without affecting the accuracy.

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