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Radial deformation and stresses in anisotropic nonhomogeneous elastic media

AMARENDRA DAS
Chandernagore Government College, Hooghly 712 136, West Bengal, India.

AND

ASA CHAKRABORTY
Chandi Bhawan, Chandernagore, Hooghly 712 136, West Bengal, India.

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Abstract

In this paper we study the elasticity problem of a cylindrically-anisotropic, elastic medium bounded by two axisymmetric cylindrical surfaces subjected to normal pressures (plane-strain). The material of the structure is orthotropic with cylindrical anisotropy and, in addition, is continuously inhomogeneous with mechanical properties varying along the radius. General solutions in terms of Whittaker functions are presented. The results obtained by St. Venant for a homogeneous cylindrically-anisotropic medium can be deduced from the general solutions. Problems of the type covered in this paper are encountered in nuclear reactor design.

Key words: Radial deformation, elastic media.

1. Introduction

The elastic behaviour of a homogeneous cylindrically aeolotropic material was first studied by St. Venant[1,2]. Problems involving nonhomogeneous media in which the properties vary continuously with spatial position have been studied by various authors. Greif and Chou[3] have adopted a numerical integration method and used computer in solving the vibration problem of a cylindrically-anisotropic nonhomogeneous cylindrical shell (plane-strain).

A plane-strain assumption is also used here to find the analytical solution for the radial deformation and corresponding stresses in a cylindrical shell made of cylindrically-aeolotropic heterogeneous material under the influence of normal pressures on both boundaries. The results obtained by St. Venant[2] for the homogeneous anisotropic case and those found by Lame'[1] for the homogeneous isotropic case can be deduced from the general results. The nonhomogeneity of the material is characterized by the elastic parameters—

e_{ij} (See refs. 3 and 4) as

$$e_{ij} = \lambda_{ij}\, r^{2\beta} \exp(-kr^{2\alpha}) \qquad (i,j = 1,2,3) \tag{1}$$

where λ_{ij}, β, k and α are the prescribed parameters of material concerned.

2. Fundamental equations

The basic system of field equations in linear isothermal static elasticity theory is (a) the generalised Hooke's Law, (b) the linearized strain displacement equations, and (c) the stress equations of equilibrium. Here the axis of anisotropy is taken to be z-axis of the r,θ,z cylindrical co-ordinate system, and the Young's moduli are of the form

$$E_r = E_1 \cdot r^{2\beta} \cdot \exp(-k \cdot r^{2\alpha})$$
$$E_\theta = E_2 \cdot r^{2\beta} \cdot \exp(-k \cdot r^{2\alpha}), \text{ etc.} \tag{2}$$

For plane-strain assumption λ_{ij} of equation (1) is then expressible in terms of E_1, E_2 and the Poisson's ratios[3].

For the axisymmetric case the non-trivial stress equation of equilibrium, in the absence of body forces, takes the form

$$\frac{\partial \bar{\sigma}_r}{\partial r} + 1/r\,(\bar{\sigma}_r - \bar{\sigma}_\theta) = 0 \tag{3}$$

Non-zero stresses in the normal, circumferential and longitudinal directions are

$$\bar{\sigma}_r = \left(\lambda_{11}\frac{d\bar{u}}{dr} + \lambda_{12}\frac{\bar{u}}{r}\right) \cdot r^{2\beta} \cdot \exp(-k.\, r^{2\alpha})$$

$$\bar{\sigma}_\theta = \left(\lambda_{12}\frac{d\bar{u}}{dr} + \lambda_{22}\frac{\bar{u}}{r}\right) \quad r^{2\beta} \cdot \exp(-k.\, r^{2\alpha})$$

$$\bar{\sigma}_z = \left(\lambda_{13}\frac{d\bar{u}}{dr} + \lambda_{23}\frac{\bar{u}}{r}\right) \cdot r^{2\beta} \cdot \exp(-k.\, r^{2\alpha}) \tag{4}$$

respectively, u being the radial displacement.

3. Method of solution

The equation of equilibrium (3), with the help of equations (4) becomes

$$\frac{d^2\bar{u}}{dr^2} + (2\beta + 1 - 2\alpha k \cdot r^{2\alpha}) \cdot r \cdot \frac{d\bar{u}}{dr} + \frac{(2\beta\lambda_{12} - \lambda_{22} - k \cdot 2\alpha \cdot r^{2\alpha} \cdot \lambda_{12})}{\lambda_{11}} \cdot \bar{u} = 0 \tag{5}$$

$$x = k \cdot r^{2\alpha} \quad \text{and} \quad \bar{u} = V \exp(x/2) \tag{6}$$

Equation (5) changes to

$$x^2 \cdot \frac{d^2 v}{dx^2} + \left(\frac{\beta}{\alpha} + 1\right) \cdot x \cdot \frac{dv}{dx} + \left[\frac{2\lambda_{12}\beta - \lambda_{22}}{4\alpha^2 \lambda_{11}} + \left\{\frac{\beta}{\alpha} + 1 - \frac{\lambda_{12}}{\alpha \cdot \lambda_{11}}\right\} \cdot \frac{x}{2} - \frac{x^2}{4}\right] \cdot V = 0 \tag{7}$$

Again for

$$V = x^{-1/2(\beta/\alpha + 1)} \cdot u \tag{8}$$

Equation (7) reduces to

$$x^2 \cdot \frac{d^2 u}{dx^2} + \left[\frac{1}{4} - \frac{\lambda_{22} - \beta\{2\lambda_{12} + (2\alpha + \beta) \cdot \lambda_{11}\}}{4\alpha^2 \lambda_{11}} \right.$$

$$\left. + \left\{ \frac{1}{2}\left(\frac{\beta}{\alpha} + 1\right) - \frac{\lambda_{12}}{2\alpha\lambda_{11}} \right\} \cdot x - x^2/4 \right] \cdot u = 0 \tag{9}$$

The solution of the above differential equation is[5].

$$U = A\, M_{k',p'}(x) + B\, M_{k',-p'}(x) \tag{10}$$

where $M_{k',\pm p'}(x)$ are Whittaker functions in which

$$2p' = \left[\frac{\lambda_{22} - \beta\{2\lambda_{12} + (2\alpha + \beta)\lambda_{11}\}}{\alpha^2 \lambda_{11}} \right]^{1/2} \tag{11}$$

$$k' = \frac{1}{2}(\beta/\alpha + 1) - \frac{\lambda_{12}}{2\alpha\lambda_{11}} \tag{12}$$

A and B being arbitrary constants.

If $2p'$ is an integer or zero, the solution of equation (9) may be written as

$$U = C\, W_{k',p'}(x) + D\, W_{-k',p'}(-x)$$

where

$$W_{k',p'}(x) = \frac{\Gamma(e-1)}{\Gamma(d-e+1)} M_{k',p'}(x) + \frac{\Gamma(1-e)}{\Gamma(\mathrm{d})} M_{k',-p'}(x) \tag{13}$$

in which $c = l \pm 2p'$ and $d = ½ - k' \pm p'$

Finally, the radial displacement $\bar{u}(r)$ satisfying the equilibrium, equation (5) is obtained with the help of equation (6), (8) and (10) as

$$\bar{u} = \frac{\exp(kr^{2\alpha})/2}{k^{1/2\ (\beta/\alpha+1)} \cdot r^{\alpha+\beta}} \quad [A M_{k',p'}(kr^{2\alpha}) + B M_{k',-p'}(kr^{2\alpha})] \tag{14}$$

This expression for $\bar{u}$ may be used in equation (4) to get the general expressions for the stresses in terms of A and B.

We now consider a cylindrical shell $a \leq r \leq b$. The structure is made of nonhomogeneous cylindrically-anisotropic material. The shell is under the influence of uniformly distributed internal and external pressures.

The boundary conditions are

$$\bar{\sigma}_r = -p_0 \quad (r = a)$$

$$\bar{\sigma}_r = -p_1 \quad (r = b) \tag{15}$$

On application of these boundary conditions in the first equation of (4) along with equation (14), one obtains two simultaneous equations involving the two unknowns A and B. Solving for A and B and inserting their values in (14) and (4), one obtains the complete solution for the radial displacement and stresses as

$$\bar{u} = \frac{\exp(kr^{2\alpha})/2}{r^{\alpha+\beta} \cdot M} \quad [\{p_1 \cdot b^{\alpha-\beta} \exp(kb^{2\alpha})/2\ \alpha_{-p1}(a) - p_0\, a^{\alpha-\beta}$$

$$\exp(ka^{2\alpha})/2\ \alpha_{-p'}(b)\}\ M_{k'p'}(kr^{2\alpha}) + \{p_0\, a^{\alpha-\beta} \exp(ka^{2\alpha})/2\, \alpha_{p'}(b) - p_1 b^{\alpha-\beta}$$

$$\exp(kb^{2\alpha})/2 \cdot \alpha_{p'}(a)\}\ M_{K',-p'}(kr^{2\alpha})],$$

$$\bar{\sigma}_r = \frac{\exp(-kr^{2\alpha})/2}{r^{\alpha-\beta} \cdot M} \quad [\{p_1 \cdot b^{\alpha-\beta} \exp(kb^{2\alpha})/2\, \alpha_{-p1}(a) - p_0\, a^{\alpha-\beta} \exp(ka^{2\alpha})/2 \cdot$$

$$\alpha_{-p'}(b)\}\ \alpha_{p'}(r) + \{p_0\, a^{\alpha-\beta} \exp(ka^{2\alpha})/2\ \alpha_{p'}(b) - p_1 b^{\alpha-\beta} \exp(2kb^{2\alpha})/2 \cdot \alpha_{p'}(a)\}$$

$$\alpha_{-p'}(r)],$$

$$\bar{\sigma}_\theta = \frac{\exp(-kr^{2\alpha})/2}{r^{\alpha-\beta} \cdot M} \quad [\{p_1 \cdot b^{\alpha-\beta} \exp(kb^{2\alpha})/2\alpha_{-p1}(a) - p_0\, a^{\alpha-\beta} \exp(ka_{2\alpha})/2$$

$$\alpha_{-p'}(b)\}\,\beta_{p'}(r) + \{p_0\, a^{\alpha-\beta} \exp(ka^{2\alpha})/2\,\alpha_{p'}(b) - p_1 b^{\alpha-\beta} \exp(2kb^{2\alpha})/2 \cdot \alpha_{p'}(a)\}$$

$$\beta_{-p'}(r)\,],$$

$$\bar{\sigma}_z = \frac{\exp(-kr^{2\alpha})/2}{r^{\alpha-\beta}\cdot M}\,[\,\{p_1 \cdot b^{\alpha-\beta} \exp(kb^{2\alpha})/2\;\alpha_{-p'}(a) - p_0\, a^{\alpha-\beta} \exp(ka^{2\alpha})/2 \cdot$$

$$\alpha_{-p'}(b)\}\,\gamma_{p'}(r) + \{p_0\, a^{\alpha-\beta} \exp(ka^{2A})/2\;\alpha_{p'}(b) - p_1\, b^{\alpha-\beta}\, \exp(kb^{2\alpha})/2 \cdot$$

$$\alpha_{p'}(a)\}\;\;\gamma_{-p'}(r)\,], \tag{16}$$

where

$$\alpha_{\pm p'}(r) = \{\lambda_{11}\,(k\alpha r^{2\alpha-1} - \alpha + \beta/r) + \lambda_{12}/r\}\cdot$$

$$M_{k',\pm p'}(kr^{2\alpha}) + 2\alpha\lambda_{11}\, kr^{2\alpha-1}\;\; M'_{k';\pm p'}(kr^{2\alpha})$$

$$\beta_{\pm p'}(r) = \{\lambda_{12}\,(k\alpha r^{2\alpha-1} - \alpha + \beta/r) + \lambda_{22}/r\}\cdot$$

$$M'_{k';\pm p'}(kr^{2\alpha}) + 2\alpha\lambda_{12}\, kr^{2\alpha-1}\; M'_{k';\pm p'}(kr^{2\alpha})$$

$$\gamma_{\pm p'}(r) = \{\lambda_{13}\,(k\alpha r^{2\alpha-1} - \alpha + \beta/r) + \lambda_{23}/r\}\cdot$$

$$M_{k',\pm p'}(kr^{2\alpha}) + 2\alpha\lambda_{13}\, kr^{2\alpha-1}\;\; M'_{k';\pm p'}(kr^{2\alpha})$$

and $M = \alpha_{p'}(a)\;\alpha_{-p'}(b) - \alpha_{-p'}(a)\cdot \alpha_{p'}(b)$ (17)

The prime indicates the derivative of the function with respect to its argument.

Stresses in a cylindrical shell ($a \leq r \leq b$) made of homogeneous cylindrically-anisotropic material, under the same boundary conditions (15), may be found from the second, third and fourth equations of (16) on letting $\beta \to 0$ and $k \to 0$ and these agree with the results obtained by St. Venant (quoted in ref. 2). For an isotropic body

$$\lambda_{11} = \lambda_{22} = \lambda + 2\mu$$
$$\lambda_{12} = \lambda_{13} = \lambda_{23} = \lambda$$

Equation (11), with the application of these relations, gives

$$2\alpha p' = \left[1-\beta\left\{\frac{2\lambda}{\lambda+2\mu}+(2\alpha+\beta)\right\}\right]^{1/2}$$

when these are used in the second, third and fourth equations of (16) along with the limits $\beta \to 0$ and $k \to 0$, one gets Lamês results given in ref. 1.

In computing those results one has to use

$$\operatorname*{Lt}_{\zeta\to 0} \frac{M_{k',p'}(\zeta)}{M_{k',-p'}(\zeta)} = 0$$

$$\operatorname*{Lt}_{\zeta\to 0} \frac{\zeta\, M'_{k,p'}(\zeta)}{M_{k,p'}(\zeta)} = 0 \quad \text{and} \quad \operatorname*{Lt}_{\zeta\to 0} \frac{\zeta M'_{k',\pm p'}(\zeta)}{M'_{k',\pm p'}(\zeta)} = 1/2 \pm p' \tag{18}$$

If the cylinder is under the action of internal pressure only, the external surface being stress-free, the stresses for such an in-homogeneous cylinder are obtained from (16) by taking $p_1 = 0$.

4. Numerical results

Stresses in the normal, circumferential and longitudinal directions are computed and plotted against non-dimensional radial co-ordinate (r/a) in figs. 1 and 2 for two types of loading systems for cylindrical shell structures made of *isotropic and anisotropic non-homogeneous* layered media or aggregate.

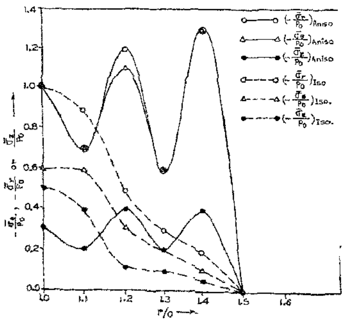

FIG. 1. Loading system I ($p_1 = 0$) isotropic and anisotropic variations.

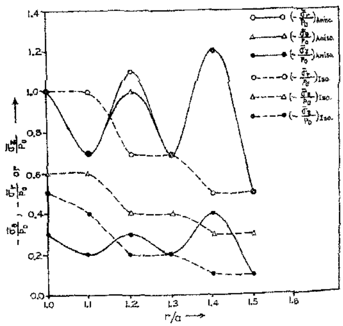

FIG. 2. Loading system II ($p_1 = p_0/2$) isotropic and anisotropic variation.

All the numerical results have been calculated for the structure whose outer radius is one and half times that of the inner radius and non-homogeneity parameters $\alpha = 2$, $\beta = -1$ and $k = 2/a^4$.

The elastic parameters are chosen as $\lambda_{11} = 918$, $\lambda_{22} = 408$, $\lambda_{12} = 918$, $\lambda_{13} = 275$, $\lambda_{23} = 273$ (see ref. (1) for anisotropic body, which resembles barite-cement aggregate, see ref. (6)), and are used extensively as radiation shielding material. On the other hand, for isotropic body, they are taken as $\lambda_{11} = \lambda_{22} = E\sigma/(1+\sigma)(1-2\sigma)$, $\lambda_{12} = \lambda_{13} = \lambda_{23} = E\sigma/\sigma(1+\sigma)$ with $\sigma = 1/3$ but E may be anything. The graphs clearly show the difference of variations in stresses for the above mentioned structures made of isotropic and anisotropic materials while they are keeping their non-homogeneous character since they are formed of layered media having variation of stiffness according to the law given in (1) or where they are formed out of aggregate. The mechanical loading system *I* corresponds to the fact that the structure is stress-free in the outer surface but the inner one is subjected to pressure p_0. In the loading system *II* the inner surface has the same pressure while at the outer surface $\bar{\sigma}_r = -p_0/2$.

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